Geometric Reasoning with Invariant Algebras

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Abstract: Geometric reasoning is a common task in Mathematics Education, Computer-Aided Design, Computer Vision and Robot Navigation. Traditional geometric reasoning follows either a logical approach in Artificial Intelligence, or a coordinate approach in Computer Algebra, or an approach of basic geometric invariants such as areas, volumes and distances. In algebraic approaches to geometric reasoning, geometric interpretation is needed for the result after algebraic manipulation, but in general this is a difficult task. It is hoped that more advanced geometric invariants can make some contribution to the problem.

In this paper, a hierarchical framework of invariant algebras is introduced for algebraic manipulation of geometric problems. The bottom level is the algebra of basic geometric invariants, and the higher levels are more complicated invariants. High level invariants can keep more geometric nature within algebraic structure. They are beneficial to geometric explanation but more difficult to handle with than low level ones. This paper focuses on geometric computing at high invariant levels, for both automated theorem proving and new theorem discovering. Using the new techniques developed in recent years for hierarchical invariant algebras, better geometric results can be obtained from algebraic computation, in addition to more efficient algebraic manipulation.

1 Background: Geometric Invariants

What is geometric computing? Computing is an algebraic thing. If the objects involved are geometric, then in some suitable algebraic framework, the geometric problem is translated into an algebraic one, and an algebraic result is obtained by computation [18]. The computing is finished only after the algebraic result is translated into geometric one. Unfortunately in many cases, the last step is not only difficult, but impossible because the result is not an invariant.

Then what is an invariant? Take the 2D geometry as an example. A point $\mathbf{i}$ in the plane is represented by its homogeneous coordinates (x_i, y_i, z_i). For projective geometry, the coordinate transformation group is the general linear group of $\mathcal{R}^3$ in which a 2D projective point is just a 1D linear subspace. Under a nondegenerate linear transform

$$\mathbf{x} = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \mapsto \mathbf{x}' = \begin{pmatrix} x_i' \\ y_i' \\ z_i' \end{pmatrix} = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \tag{1.1}$$

consider the determinant formed by the coordinates of three points $\mathbf{1}, \mathbf{2}, \mathbf{3}$, called the *bracket* of the points:

$$[\mathbf{123}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \tag{1.2}$$

Under the transform (1.1), the bracket is changed into

$$[\mathbf{1}'\mathbf{2}'\mathbf{3}'] = \det(\mathbf{A})[\mathbf{123}]. \tag{1.3}$$

A *projective invariant* c is a homogeneous polynomial of the homogeneous coordinates of a set of projective points, such that under an arbitrary general linear transform $\mathbf{A}$ of the realization vector space of the projective space, c changes by a scale which is a nonnegative integer power of the determinant of $\mathbf{A}$:

$$c' = \det(\mathbf{A})^r c. \tag{1.4}$$

Brackets are projective invariants. In fact they generate all other projective invariants algebraically [15].

Historically projective invariants are among the first invariants that people studied. In studying these classical invariants, D. Hilbert established his three famous theorems: Hilbert Nullstellensatz, Hilbert Basis Theorem, Hilbert Syzygy Theorem. In geometric computing, people used coordinate polynomials to represent and compute invariants. This approach, however, is doomed to failure because of the difficulty of translating coordinate polynomials into invariant ones. In the 1970's, the Rota school launched a new approach of invariant computing by using the symbolic representation of brackets instead of the coordinate one, and using the syzygies among the brackets, i.e., their algebraic relations, in algebraic manipulation. In the symbolic bracket approach, the computations are often so concise that each step contains only two terms, e.g., in the biquadratic final polynomial approach [17].

An affine space is a hyperplane away from the origin of the realization vector space of a projective space. An affine transform is a projective transform leaving this hyperplane invariant. Any projective invariant is also affine invariant. The homogeneous coordinates of an affine point have the property that the coordinate corresponding to the basis vector not parallel to the hyperplane is constant, say it is 1. Then bracket $[\mathbf{123}]$ is just twice the signed area of triangle $\mathbf{123}$. The sign is determined by the orientation of the triangle in the affine plane. From this we get the classical geometric interpretation of the brackets: they are signed volumes of affine simplices.

Affine geometric computing based on areas directly, instead of barycentric coordinates, can be found in a number of literatures, e.g., the area method [1]. Since the barycentric coordinates are ratios of areas in which at least two of the three basis vectors occur, the difference between the area approach and the coordinate approach does not lie in their algebraic structure, but in their algebraic manipulation. Indeed, the former uses more syzygies than the latter, and allows algebraic dependencies in their elements, while the latter eliminates all algebraic dependencies. In contrast, the former approach allows for better performance in simplifying algebraic manipulation. Sometimes it is so simple that the proofs it generated are called "geometric", meaning that there is no or little difference between the algebraic approach and the Euclidean synthetic approach [1]. The completeness of the area approach, however, is not guaranteed. The coordinate approach guarantees the completeness of algebraic manipulation with heavy computing cost.

Euclidean geometry studies the affine space with an additional distance structure. The distance is induced by a Euclidean inner product structure (it is also correct to say this inversely). In fact, the square of the distance of two points is the inner product of the displacement vector between the two points with itself, and the inner product of two arbitrary vectors is the polarization of the inner product of a vector with itself. The geometry of an inner-product space is called *orthogonal geometry*, and the geometry of a distance space is called *distance geometry*. Their difference seems little (the former has an origin while the latter does not), but since the transform groups are quite different, so are the geometric invariants and computing techniques. Both geometries occurred at least in the 1940's, and a number of subsequent work followed, e.g., [1] and [16].

In orthogonal geometry, the basic invariants are inner products and brackets. In [15] it is said that only inner products are basic invariants. This statement has no contradiction with our statement because for Euclidean invariants Sturmfels only allows $r = 0$ in (1.4). The two invariants are called "Pythagorean distances" and "signed areas" to signify their geometric meaning:

$$\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{CB}} = \frac{d^2_{\mathbf{AB}} + d^2_{\mathbf{CB}} - d^2_{\mathbf{AC}}}{2} = \frac{P_{\mathbf{ABC}}}{2}, \quad [\mathbf{ABC}] = \overrightarrow{\mathbf{AB}} \wedge \overrightarrow{\mathbf{CB}} = 2S_{\mathbf{ABC}}. \tag{1.5}$$

Having talked so far already, let us come back to the topic of this paper. The practical situation for geometric computing with invariant algebra is that using invariants simplifies algebraic description with the cost of more complicated algebraic manipulation: because of the algebraic dependencies among the basic elements, even the determination of whether or not two bracket polynomials are equal becomes a nontrivial task. Another aspect of invariant computing is that the invariants are still too low level: big bracket polynomials often occur in the procedure computation, making algebraic computation very complicated. The third aspect is that although the result of invariant computing is invariant and thus geometrically interpretable, the interpretation is still difficult to make: the result is usually a complicated rational bracket polynomial without any clear geometric meaning.

To alleviate the heavy duty of algebraic manipulation and at the same time keep more geometric nature within the algebraic structure, a hierarchy of high-level invariants is needed: the bottom level is the basic invariants, and each higher level invariant is a polynomial of the lower level ones. By putting the high level invariants into the algebra of basic invariants as new indeterminates, and treating their algebraic relations with basic ones as defining syzygies, we obtain an algebra of advanced invariants.

For example, in the 2D projective plane there are three pairs of lines $(\mathbf{12}, \mathbf{34})$, $(\mathbf{1'2'}, \mathbf{3'4'})$, and $(\mathbf{1''2''}, \mathbf{3''4''})$. They intersect at points $\mathbf{5}, \mathbf{5'}, \mathbf{5''}$ respectively. To compute bracket $[\mathbf{55'5''}]$ one only needs to substitute the expression of each intersection by the two points on a line of the line pair, then expand the result. In homogeneous coordinates there are $2^3 = 8$ different (although equal) results, depending on the selection of one of the two lines in each line pair for the representation of the intersection. In bracket algebra, however, there are 16,847 different results [10]. When the high level invariant

$$[(\mathbf{12} \vee \mathbf{34})(\mathbf{1'2'} \vee \mathbf{3'4'})(\mathbf{1''2''} \vee \mathbf{3''4''})] \tag{1.6}$$

is used to represent the intersection, it unifies 16,847 different forms of basic invariants. The advanced invariant is not only geometrically meaningful, but also simplifies syzygy manipulations through its structural symmetries.

In this paper we shall concentrate on a particular advanced invariant algebra: null bracket algebra [9]. It is suitable for geometric computing of all classical geometries: projective, affine, Euclidean, spherical, non-Euclidean, conformal, etc. Its covariant companion is *conformal geometric algebra*, which is a universal algebraic framework for classical geometries [2], [14]. Conformal geometric algebra is established on the homogeneous version of the classical conformal model. So we first make an introduction of this model.

2 The Homogeneous Model and Conformal Geometric Algebra

We live in a 3D physical space with Euclidean structure. The usual algebraic description of the space is by means of a Cartesian coordinate system, which is composed of a special point: the origin, and three linearly independent vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, such that the inner product of $\mathbf{e}_i$ and $\mathbf{e}_j$ is 1 if $i = j$, and 0 otherwise. This is the *Cartesian model* of 3D space.

The problem with the Cartesian model is that points and directions cannot be distinguished from each other. Both are represented by vectors. Although the former are represented by vectors starting from the origin while the latter by free vectors that can move in a parallel manner any where in the space, algebraically they have no difference. As a consequence, the Euclidean transforms, or more generally affine transforms, do not have linear representation.

When the origin of the coordinate system is dropped out of the physical space, we enter a realm of 4D space, in which the 3D physical space is a hyperplane away from the origin. This is the *homogeneous coordinate model* of the 3D space. In fact it is the model for 3D projective space, and each 1D subspace of the 4D linear space is a projective point. Points in the 3D physical space are intersections of the projective points with the 3D space. In homogeneous coordinates, they have the same coordinate in the basis direction not parallel to the 3D space.

A nice property of the homogeneous coordinate model is that points and directions are separated algebraically: points correspond to vectors not parallel to the 3D affine space, while directions are parallel to the 3D space. As a consequence, affine transforms are represented by linear transforms. This property has further consequences. Grassmann established his famous *extension theory* on this model. In this theory, a point or direction is a grade one object, two points extend via their outer product to a line incident to them, and the result is a grade two object; three points extend via the outer product to a plane incident to them, and the result is a grade three object, etc. *Grassmann algebra* is the result of this extension: it is the linear space spanned by the extension results, called *Grassmann space*, equipped with the outer product. The algebraic grade structure corresponds naturally to the geometric dimension structure.

For 3D projective geometry and affine geometry, the 4D homogeneous coordinate model is sufficient. But there is still the Euclidean structure left. The 3D linear subspace parallel to the 3D affine space is the space of displacements, also called the space of points at infinity, or directions. The Euclidean inner product is defined in this subspace. Thus in the homogeneous coordinate model, the inner product is not globally defined. If would be nice if the inner product can be extended to the whole 4D space. However, any extension with clear geometric meaning is impossible.

For example, let $\mathbf{1}, \mathbf{2}, \mathbf{3}$ be three points in the affine space. Then the following inner product

$$\overrightarrow{\mathbf{12}} \cdot \overrightarrow{\mathbf{32}} = (\mathbf{2} - \mathbf{1}) \cdot (\mathbf{2} - \mathbf{3}) \tag{2.7}$$

is meaningful. Since the inner product is bilinear, one may consider expanding the right side of (2.7) to get

$$\mathbf{2}\cdot\mathbf{2}+\mathbf{2}\cdot\mathbf{3}-\mathbf{1}\cdot\mathbf{2}+\mathbf{1}\cdot\mathbf{3}, \tag{2.8}$$

however the $\mathbf{i}\cdot\mathbf{j}$'s are geometrically meaningless. In other words, the 4D surrounding space does not have an inner product structure compatible with the Euclidean geometry of the 3D subspace such that the inner products are geometrically meaningful.

In the conformal model [13], nD Euclidean geometry is realized in $(n+2)$D Minkowski space, or more accurately, in the following subset:

$$\mathcal{N}_{\mathbf{e}} = \{\mathbf{x}\in\mathcal{R}^{n+1,1}\,|\,\mathbf{x}\cdot\mathbf{x}=0,\ \mathbf{x}\cdot\mathbf{e}=-1\}. \tag{2.9}$$

Here $\mathbf{e}$ is a null vector in the $(n+2)$D Minkowski space, i.e., $\mathbf{e}\cdot\mathbf{e}=0$.

Elements in $\mathcal{N}_{\mathbf{e}}$ are in one-to-one correspondence with points in nD Euclidean space. Let null vector $\mathbf{e}_0\in\mathcal{N}_{\mathbf{e}}$ be the origin. In the conformal model, a point $\mathbf{x}$ in $\mathcal{R}^n$ is represented by the null vector

$$\mathbf{x}' = \mathbf{e}_0+\mathbf{x}+\frac{\mathbf{x}^2}{2}\mathbf{e}. \tag{2.10}$$

For $\mathbf{x}'_i = \mathbf{e}_0+\mathbf{x}_i+\frac{\mathbf{x}_i^2}{2}\mathbf{e}$,

$$|\mathbf{x}_i-\mathbf{x}_j| = |\mathbf{x}'_i-\mathbf{x}'_j|, \tag{2.11}$$

so $\mathcal{N}_{\mathbf{e}}$ is an isometric model. As a consequence, the inner product in $\mathcal{N}_{\mathbf{e}}$ is geometrically meaningful: for points $\mathbf{a},\mathbf{b}$ and their null vector representations $\mathbf{a}',\mathbf{b}'$,

$$\mathbf{a}'\cdot\mathbf{b}' = -\frac{|\mathbf{a}'-\mathbf{b}'|^2}{2} = -\frac{d_{\mathbf{ab}}^2}{2}, \tag{2.12}$$

The conformal model is not only a non-Euclidean model for Euclidean geometry, but also a quadratic model for affine geometry: the 3D geometric space is not realized as an affine space, but a quadric space, in the 5D Minkowski space. The model has its geometric origin in the work of Wachter (1840's), and algebraically it is formally stated in the Ph. D. thesis of S. Lie (1872). It has the very nice feature that (1) the grade structure of the Minkowski space fits naturally into the geometric structure of the 3D geometry: points are grade one objects, 0D lines and circles (point pairs) are grade two objects, 1D lines and circles are grade three objects, 2D planes and spheres are grade four objects, etc.; (2) the inner product of the Minkowski space has clear Euclidean geometric meaning when restricted to the set of algebraic elements representing geometric objects: the inner product of two points or circles, or a point and a circle, is their squared Euclidean distance; the inner product of two lines, or a point and a line, or a line and a circle, is the cosine of the angle of intersection if they intersect, or the signed distance between them in other cases [3].

From the definition (2.9), it is clear that the conformal model depends on the choice of the origin $\mathbf{e}_0$. In [12], a more general version is proposed, called the *homogeneous model*. It is based on the set of null vectors

$$\mathcal{N} = \{\mathbf{x}\in\mathcal{R}^{n+1,1}\,|\,\mathbf{x}\cdot\mathbf{x}=0\} \tag{2.13}$$

and a null vector $\mathbf{e}\in\mathcal{N}$ called the *point at infinity*. An element $\mathbf{x}\in\mathcal{N}$ represents a point if and only if $\mathbf{x}\cdot\mathbf{e}\neq 0$. Two elements in $\mathcal{N}$ represents the same point if and only if they differ by a nonzero scale. This representation is homogeneous, and the model is conformal instead of isometric. Because of this, it can represent various classical geometries of different metrics [14].

On the other hand, with the introduction of two additional dimensions, algebraic computation based on coordinates in the $(n+2)$D space becomes more complicated, and using invariants in this framework becomes more important.

The geometric algebra established upon the homogeneous model is called *conformal geometric algebra* [5]. It is the covariant algebra of Euclidean distance geometry. For invariant computation, conformal geometric algebra provides a natural setting for finding advanced geometric invariants. We shall leave the discussion of advanced invariants for the next session, and concentrate on the system of basic invariants instead.

The algebra of basic invariants in the conformal model is the inner-product bracket algebra in which all vectors are null. There are four kinds of basic invariants:

1. $\mathbf{e} \cdot \mathbf{a}$, where $\mathbf{a}$ is a null vector representing a point. In the conformal setting,

$$\mathbf{e} \cdot \mathbf{a} = -1. \tag{2.14}$$

2. $\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ are null vectors representing two points. In the conformal setting,

$$\mathbf{a} \cdot \mathbf{b} = -\frac{d_{\mathbf{ab}}}{2}. \tag{2.15}$$

3. $[\mathbf{e12} \cdots (\mathbf{n+1})]$, where null vectors $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{n+1}$ represent different points in the nD Euclidean space.

   In the conformal setting, when $n = 2$, $[\mathbf{e123}]$ equals twice the signed area of triangle $\mathbf{123}$. In particular, $[\mathbf{e123}] = 0$ if and only if $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are collinear. For general n, the bracket is $1/n!$ times the signed volume of the simplex with vertices $\mathbf{1}$ to $\mathbf{n+1}$.

4. $[\mathbf{12} \cdots (\mathbf{n+2})]$, where null vectors $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{n+2}$ represent different points in the nD Euclidean space.

   In the conformal setting, when $n = 2$, then if points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ are collinear, then $[\mathbf{1234}] = 0$, else if $\mathbf{123}$ is a triangle, let $\mathbf{0}$ be its circumcenter and r be its radius, then

$$[\mathbf{1234}] = S_{\mathbf{123}}(d_{\mathbf{04}}^2 - r^2). \tag{2.16}$$

   In particular, $[\mathbf{1234}] = 0$ if and only if $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ are cocircular or collinear.

   For general n, on the right side of (2.16), $\mathbf{0}$ is the circumcenter of the simplex generated by the first $n + 1$ points in the sequence, and S is the signed volume of this simplex.

In the above list, the third is just the classical bracket of $n + 1$ points in nD projective geometry. The first is the coordinate corresponding to the basis vector not parallel to the hyperplane representing the nD Euclidean (affine) space. The first and the third are all the basic affine invariants, and the third is all the basic projective invariants.

3 Algebra of Advanced Invariants

First we consider projective geometry. The covariant algebra of projective geometry is called *Grassmann-Cayley algebra* [16]. This is an algebra equipped with two multiplications which are dual to each other: the outer product, usually denoted in invariant theory by the juxtaposition of elements instead of the wedge symbol, and the meet product, denoted by the symbol "$\vee$". The dual operator is defined only for elements of grade $n+1$, where n is the dimension of the projective space. In fact, the dual of the outer product of $n+1$ vectors is just the bracket formed by them.

An expression in Grassmann-Cayley algebra, called *Cayley expression*, refers to a polynomial (noncommutative!) formed by a set of vectors through the two products. All scalar-valued expressions and the dual of grade-$(n+1)$-valued expressions generate a polynomial ring (commutative!), called the *Cayley bracket algebra*. The generating elements are naturally graded by the number of products of vectors, and any monomial in this algebra has clear geometric meaning: the outer product represents the geometric extension, and the meet product represents the geometric intersection. This is an algebra of advanced projective invariants, and includes the bracket algebra as a subalgebra.

We use an example of automated theorem proving in nD projective geometry to show the role of advanced invariants play in representing and computing geometric relations. The *Desargues Theorem* and its inverse in nD projective space is stated as follows:

Let there be two triangles $\mathbf{123}$ *and* $\mathbf{1'2'3'}$ *in* n*D projective space. Assume that the three pairs of corresponding edges intersect at points*

$$\mathbf{a} = \mathbf{12} \cap \mathbf{1'2'}, \quad \mathbf{b} = \mathbf{23} \cap \mathbf{2'3'}, \quad \mathbf{c} = \mathbf{13} \cap \mathbf{1'3'}$$

respectively. Then points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are collinear if and only if lines* $\mathbf{11'}, \mathbf{22'}, \mathbf{33'}$ *concur.*

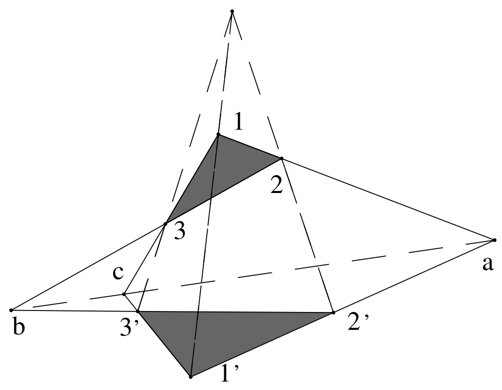

Figure 1: Desargues Theorem in nD space.

In the $(n+1)$D vector space realizing the nD projective space, the intersection of lines $\mathbf{12}$ and $\mathbf{1'2'}$ is just the intersection of line $\mathbf{12}$ and hyperplane $\mathbf{1'2'U}$, where $\mathbf{U}$ is the outer product of $(n-2)$ vectors in generic position and thus represents a generic $(n-3)$D plane. The reason for this "extension" is that in nD geometry generally two lines do not intersect, while a line and a hyperplane do. Extending line $\mathbf{1'2'}$ to a generic hyperplane passing through this line simplifies algebraic manipulation considerably.

There is a deeper reason for this extension. The intersection of two lines in nD geometry cannot be represented by algebraic invariants which usually refer to polynomials, but only by *rational invariants*. A rational invariant is a rational polynomial of coordinates which is invariant as a whole but neither the numerator nor the denominator of which is invariant.

In Grassmann-Cayley algebra, $\mathbf{a} = \mathbf{12} \cap \mathbf{1'2'}$ is represented by

$$\mathbf{a} = \mathbf{12} \vee \mathbf{1'2'U} = [\mathbf{11'2'U}]\mathbf{2} - [\mathbf{21'2'U}]\mathbf{1}. \tag{3.17}$$

The second equality is the *expansion* from Grassmann-Cayley algebra to bracket algebra, or more generally, from the algebra of covariants to the algebra of invariants.

When the meaning of the symbols is clear from the context, the dummy element $\mathbf{U}$ is often omitted, and (3.17) becomes

$$\mathbf{a} = \mathbf{12} \vee \mathbf{1'2'} = [\mathbf{11'2'}]\mathbf{2} - [\mathbf{21'2'}]\mathbf{1}. \tag{3.18}$$

Since the representation of a projective point by a vector is unique only up to a nonzero scale, (3.18) can be written in terms of rational invariants:

$$\mathbf{a} \simeq \mathbf{2} - \frac{[\mathbf{21'2'}]}{[\mathbf{11'2'}]}\mathbf{1}$$

where "$\simeq$" denotes that the two sides of the symbol are equal up to a nonzero scale.

In nD projective space, points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear if and only if $\mathbf{a}$ is on the hyperplane $\mathbf{bcU}$, i.e.,

$$[\mathbf{abcU}] = 0. \tag{3.19}$$

When dummy element $\mathbf{U}$ is omitted, the equation becomes $[\mathbf{abc}] = 0$. Similarly, in nD space three lines $\mathbf{11'}, \mathbf{22'}, \mathbf{33'}$ concur if and only if line $\mathbf{11'}$, hyperplane $\mathbf{22'U}$ and hyperplane $\mathbf{33'U}$ concur, i.e.,

$$\mathbf{11'} \vee \mathbf{22'U} \vee \mathbf{33'U} = [\mathbf{122'U}][\mathbf{1'33'U}] - [\mathbf{1'22'U}][\mathbf{133'U}] = 0. \tag{3.20}$$

When dummy element $\mathbf{U}$ is omitted, the equation becomes

$$\mathbf{11'} \vee \mathbf{22'} \vee \mathbf{33'} = [\mathbf{122'}][\mathbf{1'33'}] - [\mathbf{1'22'}][\mathbf{133'}] = 0. \tag{3.21}$$

Thus, by means of rational invariants, the incidence relations of points and lines in nD projective geometry have the same algebraic representations as long as $n > 1$. nD Desargues Theorem states that if intersections $\mathbf{a}, \mathbf{b}, \mathbf{c}$ exist, then $[\mathbf{abc}] = 0$ if and only if $\mathbf{11'} \vee \mathbf{22'} \vee \mathbf{33'} = 0$. This theorem is a direct corollary of the following algebraic identity:

$$[(\mathbf{12} \vee \mathbf{1'2'})(\mathbf{23} \vee \mathbf{2'3'})(\mathbf{31} \vee \mathbf{3'1'})] = [\mathbf{123}][\mathbf{1'2'3'}]\mathbf{11'} \vee \mathbf{22'} \vee \mathbf{33'}. \tag{3.22}$$

The proof of (3.22) is very easy. First expand the meet products into brackets:

$$\begin{aligned}
\mathbf{12} \vee \mathbf{1'2'} &= [\mathbf{11'2'}]\mathbf{2} - [\mathbf{21'2'}]\mathbf{1}, \\
\mathbf{23} \vee \mathbf{2'3'} &= [\mathbf{22'3'}]\mathbf{3} - [\mathbf{32'3'}]\mathbf{2}, \\
\mathbf{31} \vee \mathbf{3'1'} &= [\mathbf{11'3'}]\mathbf{3} - [\mathbf{31'3'}]\mathbf{1},
\end{aligned}$$

Then the left side of (3.22) becomes

$$[(\mathbf{12} \vee \mathbf{1'2'})(\mathbf{23} \vee \mathbf{2'3'})(\mathbf{31} \vee \mathbf{3'1'})] = [\mathbf{123}]([\mathbf{11'3'}][\mathbf{21'2'}][\mathbf{32'3'}] - [\mathbf{11'2'}][\mathbf{22'3'}][\mathbf{31'3'}]), \tag{3.23}$$

Second (and last), do Cayley factorization [10] to the polynomial factor of (3.23) to finish the proof:

$$[\mathbf{11'3'}][\mathbf{21'2'}][\mathbf{32'3'}] - [\mathbf{11'2'}][\mathbf{22'3'}][\mathbf{31'3'}] = [\mathbf{1'2'3'}]\mathbf{11'} \vee \mathbf{22'} \vee \mathbf{33'}. \tag{3.24}$$

(3.22) says that from the incidence of three lines one obtains the collinearity of three intersections, but only under the nondegeneracy conditions that neither $\mathbf{1}, \mathbf{2}, \mathbf{3}$ nor $\mathbf{1'}, \mathbf{2'}, \mathbf{3'}$ are collinear can one deduce the incidence of three lines from the collinearity of the intersections.

In comparison, for the case $n = 3$, the coordinate polynomial form of the right side of (3.22) has 1290 terms, and after factorization, the biggest factor has 48 terms. The geometric meaning is hard to find out. By means of invariants, in particular advanced invariants, both the efficiency and geometric effectivity of algebraic computing are greatly improved.

Next we consider nD Euclidean geometry. Since conformal geometric algebra is the covariant algebra of this geometry, the advanced invariants should be generated by the product of this algebra, called the *geometric product*, or *Clifford multiplication* [6]. It is the unique multilinear product such that the product of any two vectors equals their inner product. The polynomial system of invariants generated in this way is the *Clifford bracket algebra* [7], whose monomials are scalar-valued expressions and the dual of grade-$(n+2)$-valued expressions in geometric algebra. In the conformal setting, all vectors are null if they represent points, and the corresponding Clifford bracket algebra is called *null bracket algebra*.

There are three kinds of basic advanced invariants in Clifford bracket algebra:

- Angular bracket $\langle \mathbf{12} \cdots (\mathbf{2k}) \rangle$, where $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{2k}$ are null vectors representing a sequence of $2k$ points including the point at infinity, and the same point can occur multiple times.

  When $k = 1$, then $\langle \mathbf{12} \rangle$ is the inner product of vectors $\mathbf{1}, \mathbf{2}$; for general k, the angular bracket is the scalar part of the geometric product of the sequence of $2k$ vectors.

- Parenthesis (also called Gram determinant, or bideterminant) $(\mathbf{12} \cdots \mathbf{r} \,|\, \mathbf{1'2'} \cdots \mathbf{r'})$, where $1 \leq r \leq n+2$, and $\mathbf{1}, \mathbf{1'}, \cdots, \mathbf{r}, \mathbf{r'}$ are null vectors.

  In fact,
  $$(\mathbf{12} \cdots \mathbf{r} \,|\, \mathbf{1'2'} \cdots \mathbf{r'}) = \det(\mathbf{i} \cdot \mathbf{j'})_{i,j=1..r}.$$

- Square bracket $[\mathbf{12} \cdots (\mathbf{n} + \mathbf{2k})]$, where $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{n} + \mathbf{2k}$ are null vectors representing points that may be identical, including the point at infinity.

  When $k = 1$, $[\mathbf{12} \cdots (\mathbf{n} + \mathbf{2})]$ is the classical bracket (determinant) of the $n+2$ vectors; for general k, the square bracket is the dual of the grade-$(n+2)$ part of the geometric product of the sequence of vectors.

Prolonging the length of elements in inner products and (classical) brackets is not only a device to make advanced geometric computing, but an indispensable means in representing basic geometric relations. We take a look at a planar angle and its algebraic representation.

An oriented angle is just a 2D rotation. $\angle\mathbf{123}$ can be represented by three points: the vertex $\mathbf{2}$, a point $\mathbf{1}$ on the initial ray, and a point $\mathbf{3}$ on the terminal ray. Generally it is not required that $\mathbf{1}, \mathbf{3}$ be equi-distant from $\mathbf{2}$. The following are plain facts about planar angles:

- The angle itself is transcendental with respect to the coordinates of the three points.
- The sine and cosine of the angle is irrational with respect to the coordinates. However, when multiplied by the lengths of the two edges, they become polynomials of the coordinates, which are the inner product and outer product of the two edge vectors.
- For 2D geometry, the tangent function of the angle is a rational function of the coordinates. It describes the angle accurately up to $k\pi$.

Without resorting to inequalities, any algebraic description of an angle is accurate only up to $k\pi$. The equivalent classes of planar angles modulo π are called *full-angles* [1], [18].

Let $\angle\mathbf{123}, \angle\mathbf{1'2'3'}$ be two full angles. They are equal if and only if $\tan\angle\mathbf{123} = \tan\angle\mathbf{1'2'3'}$. In the conformal setting, the ratio of the square bracket and angular bracket of the sequence $\mathbf{e212'}$ is exactly $\tan\angle\mathbf{212'}$. So $\angle\mathbf{123} = \angle\mathbf{1'2'3'}$ if and only if

$$\frac{[\mathbf{e123}]}{\langle\mathbf{e123}\rangle} = \frac{[\mathbf{e1'2'3'}]}{\langle\mathbf{e1'2'3'}\rangle}, \tag{3.25}$$

i.e.,

$$[\mathbf{e123}]\langle\mathbf{e1'2'3'}\rangle - \langle\mathbf{e123}\rangle[\mathbf{e1'2'3'}] = 0. \tag{3.26}$$

The left side of (3.26) equals half of $[\mathbf{e123e3'2'1'}]$. It is the rational representation of the sine of the difference of the two angles. So the two full-angles are equal if and only if

$$[\mathbf{e123e3'2'1'}] = 0. \tag{3.27}$$

One sees immediately that the long bracket $[\mathbf{e123e3'2'1'}]$ is needed to represent the trigonometric functions of the difference of two angles, and hence the equality of two full-angles. Indeed, we have the following geometric explanation of the square and angular brackets in the conformal model for 2D Euclidean geometry [5]: For affine points in the plane,

$$\begin{array}{rcl} \langle\mathbf{1234}\rangle &=& -\dfrac{d_{\mathbf{12}}d_{\mathbf{23}}d_{\mathbf{34}}d_{\mathbf{41}}}{2}\cos\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}), \\ [\mathbf{1234}] &=& -\dfrac{d_{\mathbf{12}}d_{\mathbf{23}}d_{\mathbf{34}}d_{\mathbf{41}}}{2}\sin\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}), \\ \langle\mathbf{123456}\rangle &=& -\dfrac{d_{\mathbf{12}}d_{\mathbf{23}}d_{\mathbf{34}}d_{\mathbf{45}}d_{\mathbf{56}}d_{\mathbf{61}}}{2}\cos(\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}) + \angle(\overrightarrow{\mathbf{145}}, \overrightarrow{\mathbf{156}})), \\ [\mathbf{123456}] &=& -\dfrac{d_{\mathbf{12}}d_{\mathbf{23}}d_{\mathbf{34}}d_{\mathbf{45}}d_{\mathbf{56}}d_{\mathbf{61}}}{2}\sin(\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}) + \angle(\overrightarrow{\mathbf{145}}, \overrightarrow{\mathbf{156}})), \end{array} \tag{3.28}$$

and for the general case,

$$\begin{aligned}
\langle \mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{2l+2}\rangle &= -\frac{d_{\mathbf{i}_1\mathbf{i}_2}d_{\mathbf{i}_2\mathbf{i}_3}\cdots d_{\mathbf{i}_{2l+1}\mathbf{i}_{2l+2}}d_{\mathbf{i}_{2l+2}\mathbf{i}_1}}{2}\cos(\angle(\overrightarrow{\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3},\overrightarrow{\mathbf{i}_1\mathbf{i}_3\mathbf{i}_4}) \\
&\quad +\angle(\overrightarrow{\mathbf{i}_1\mathbf{i}_4\mathbf{i}_5},\overrightarrow{\mathbf{i}_1\mathbf{i}_5\mathbf{i}_6})+\cdots+\angle(\overrightarrow{\mathbf{i}_1\mathbf{i}_{2l}\mathbf{i}_{2l+1}},\overrightarrow{\mathbf{i}_1\mathbf{i}_{2l+1}\mathbf{i}_{2l+2}})); \\
[\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{2l+2}] &= -\frac{d_{\mathbf{i}_1\mathbf{i}_2}d_{\mathbf{i}_2\mathbf{i}_3}\cdots d_{\mathbf{i}_{2l+1}\mathbf{i}_{2l+2}}d_{\mathbf{i}_{2l+2}\mathbf{i}_1}}{2}\sin(\angle(\overrightarrow{\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3},\overrightarrow{\mathbf{i}_1\mathbf{i}_3\mathbf{i}_4}) \\
&\quad +\angle(\overrightarrow{\mathbf{i}_1\mathbf{i}_4\mathbf{i}_5},\overrightarrow{\mathbf{i}_1\mathbf{i}_5\mathbf{i}_6})+\cdots+\angle(\overrightarrow{\mathbf{i}_1\mathbf{i}_{2l}\mathbf{i}_{2l+1}},\overrightarrow{\mathbf{i}_1\mathbf{i}_{2l+1}\mathbf{i}_{2l+2}})).
\end{aligned} \tag{3.29}$$

Here $\overrightarrow{\mathbf{123}}$ denotes the oriented circle through points $\mathbf{1},\mathbf{2},\mathbf{3}$, and $\angle(\overrightarrow{\mathbf{123}},\overrightarrow{\mathbf{134}})$ is the angle of rotation from the tangent direction of oriented circle $\overrightarrow{\mathbf{123}}$ to the tangent direction of oriented circle $\overrightarrow{\mathbf{134}}$ at their intersection.

Below we raise an example of automated discovering of new geometric theorem to show the essential role played by the long brackets. The scenario is as follows: for a geometric theorem, if after one or several of its hypotheses are removed, the conclusion can be written as $f = 0$ in which f is a monomial in a suitable invariant algebra, then a new theorem is obtained, called the geometric *completion* of the original theorem [4]. If furthermore f is expanded as a polynomial in an invariant algebra such that each term contains a factor g_i with the property that $g_i = 0$ represents one of the removed hypotheses, and every removed hypothesis occurs once and only once as a factor in the expansion of f, this expansion is a geometric *decomposition* of the original theorem. When only one hypothesis is removed, a geometric completion is the same with a geometric decomposition, and we call it geometric *factorization* [9].

Geometric decomposition is to explore the quantitative and geometric dependency of the conclusion of a theorem on its hypotheses. Geometric completion is to generalize an existing theorem by reducing its hypotheses. In the following example, geometric completion is found out first, then geometric decomposition is obtained by expansion. The original theorem is very easy (Figure 2a):

In the plane two circles intersect at points $\mathbf{1},\mathbf{1}'$ *respectively. Draw two secant lines through them, which intersect the two circles at points* $\mathbf{2},\mathbf{3}$ *and* $\mathbf{2}',\mathbf{3}'$ *respectively, then* $\mathbf{22}'//\mathbf{33}'$.

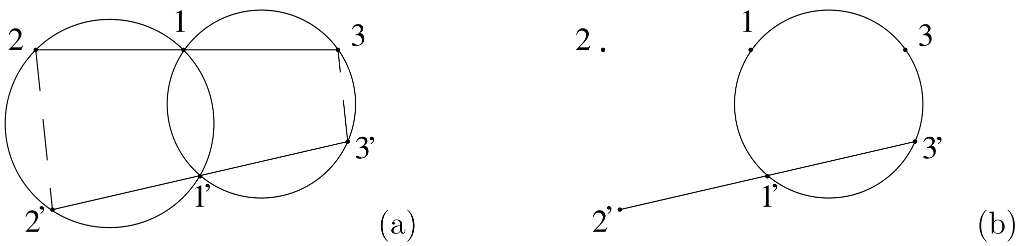

Figure 2: (a): The original theorem; (b): Two hypotheses are removed.

Now we keep only circle $\mathbf{131'3'}$ and line $\mathbf{1'2'3'}$. The new configuration (Figure 2b) has two constraints: cocircularity $[\mathbf{121'2'}] = 0$ in the homogeneous model, and collinearity $[\mathbf{e123}] = 0$ in the same model. The conclusion of the original theorem is the parallelity of two lines: $[\mathbf{e22'e33'}] = 0$.

The new configuration can be constructed as follows: let $\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{1}',\mathbf{2}'$ be free points in the plane, let $\mathbf{3}'$ be the intersection of line $\mathbf{1'2'}$ and circle $\mathbf{131'}$ other than $\mathbf{1}'$.

Point $\mathbf{3'}$ has the following expression:

$$\mathbf{3'} = \mathbf{1'13} \cap \mathbf{1'e2'} = -\mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{131'2'}]\mathbf{1'} + \frac{1}{2}[\mathbf{1'131'e2'}]([\mathbf{131'2'}]\mathbf{e} + [\mathbf{e131'}]\mathbf{2'}). \tag{3.30}$$

Substitute it into $[\mathbf{e22'e33'}]$, expand and simplify, see if the two missing hypotheses show up:

$$\begin{aligned}
[\mathbf{e22'e33'}] &= -\mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{131'2'}][\mathbf{e22'e31'}] + \frac{1}{2}[\mathbf{1'131'e2'}][\mathbf{e131'}][\mathbf{e22'e32'}] \\
&= \mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}](-[\mathbf{131'2'}][\mathbf{e22'e31'}] + [\mathbf{e232'}][\mathbf{1'131'e2'}]) \\
&= 2(\mathbf{e}\cdot\mathbf{2'})[\mathbf{e131'}](\mathbf{e}\cdot\mathbf{3}[\mathbf{e21'2'}][\mathbf{131'2'}] + \mathbf{1'}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{e232'}]) \\
&= \mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{e31'2'}][\mathbf{e311'2'2}].
\end{aligned} \tag{3.31}$$

The second and third steps of (3.31) are expansions of long brackets according to their definition, the last step is a factorization based on the following syzygy in null bracket algebra:

$$\mathbf{2}\cdot\mathbf{3}[\mathbf{122'3'}][\mathbf{31'2'3'}] + \mathbf{2'}\cdot\mathbf{3'}[\mathbf{1233'}][\mathbf{231'2'}] = -\frac{1}{2}[\mathbf{1231'2'3'}][\mathbf{232'3'}]. \tag{3.32}$$

The result is not what we expected. The two brackets $[\mathbf{e123}]$ and $[\mathbf{121'2'}]$ representing the two missing hypotheses are not in the last line of (3.31). If we expand the long bracket $[\mathbf{e311'2'2}]$, then we get the shortest expansion that is composed of six terms, with only two terms containing the missing brackets:

$$[\mathbf{e311'2'2}] = \mathbf{e}\cdot\mathbf{3}[\mathbf{121'2'}] + \mathbf{e}\cdot\mathbf{1}[\mathbf{231'2'}] + \mathbf{1}\cdot\mathbf{3}[\mathbf{e21'2'}] + \mathbf{1'}\cdot\mathbf{2'}[\mathbf{e123}] + \mathbf{2}\cdot\mathbf{1'}[\mathbf{e132'}] - \mathbf{2}\cdot\mathbf{2'}[\mathbf{e131'}]. \tag{3.33}$$

At this stage we need the *rational expansion* of the long bracket instead of the polynomial expansion. The following rational expansion has two terms, each contains a missing bracket:

$$\frac{1}{2}[\mathbf{e311'2'2}] = \frac{\mathbf{1}\cdot\mathbf{3}[\mathbf{e232'}][\mathbf{121'2'}] - \mathbf{2}\cdot\mathbf{2'}[\mathbf{e123}][\mathbf{131'2'}]}{[\mathbf{1232'}]}. \tag{3.34}$$

It seems that all computing is finished by now. Hold on, in (3.31) the left side of the first line contains a vector variable $\mathbf{3'}$, which disappears in the last line, so this is not a strict equality, but an equality in which the two sides differ by an arbitrary nonzero scale. We need to fill out this blank.

To get a strict equality, $\mathbf{3'}$ should not occur on the left side. Since it has already occurred once, we need to divide the left side by a monomial linear with respect to $\mathbf{3'}$, and follow the same computing procedure to eliminate $\mathbf{3'}$ from it. This procedure is called *homogenization*.

The simplest monomial with the required property is $\mathbf{e}\cdot\mathbf{3'}$, which is nonzero as long as $\mathbf{3'}$ is a point. Eliminating $\mathbf{3'}$ from $\mathbf{e}\cdot\mathbf{3'}$ using (3.30), we get

$$\mathbf{e}\cdot\mathbf{3'} = (\mathbf{e}\cdot\mathbf{2'})(\mathbf{1'}\cdot\mathbf{2'})[\mathbf{e131'}]^2. \tag{3.35}$$

Then

$$\frac{[\mathbf{e22'e33'}]}{\mathbf{e}\cdot\mathbf{3'}} = \frac{[\mathbf{e31'2'}][\mathbf{e311'2'2}]}{\mathbf{1'}\cdot\mathbf{2'}\,[\mathbf{e131'}]}. \tag{3.36}$$

(3.36) is the geometric completion of the original theorem under the constraints that $\mathbf{1}, \mathbf{3}, \mathbf{1}', \mathbf{3}'$ are cocircular and $\mathbf{1}', \mathbf{2}', \mathbf{3}'$ are collinear. The geometric decomposition is derived from (3.36) by rational expansion (3.34):

$$\frac{[\mathbf{e22'e33'}]}{2\,\mathbf{e}\cdot\mathbf{3}'} = \frac{\mathbf{1}\cdot\mathbf{3}[\mathbf{e232'}][\mathbf{e31'2'}]}{\mathbf{1}'\cdot\mathbf{2}'\,[\mathbf{e131'}][\mathbf{1232'}]}[\mathbf{121'2'}] - \frac{\mathbf{2}\cdot\mathbf{2}'[\mathbf{131'2'}][\mathbf{e31'2'}]}{\mathbf{1}'\cdot\mathbf{2}'\,[\mathbf{e131'}][\mathbf{1232'}]}[\mathbf{e123}]. \tag{3.37}$$

The geometric meaning of (3.36) is immediate from its components. It says

$$\overrightarrow{\mathbf{22'}} \wedge \overrightarrow{\mathbf{33'}} = \frac{2\,S_{\mathbf{21'2'}}S_{\mathbf{31'2'}}\sin(\angle(\overrightarrow{\mathbf{31}}, \overrightarrow{\mathbf{11'}}) + \angle(\overrightarrow{\mathbf{1'2'}}, \overrightarrow{\mathbf{2'2}}))}{d^2_{\mathbf{1'2'}}}. \tag{3.38}$$

Here $\overrightarrow{\mathbf{22'}} \wedge \overrightarrow{\mathbf{33'}}$ is the signed length of the vector product of vectors $\overrightarrow{\mathbf{22'}}, \overrightarrow{\mathbf{33'}}$ with respect to the unit normal of the (oriented) plane.

4 Conclusion

To alleviate the heavy task of algebraic manipulation for geometric meaningful results, invariants are introduced into geometric computation. In algebraic algebraic representation of geometric relations, advanced invariants and rational invariants occur naturally. On this paper, a hierarchy framework of invariant algebras is constructed using Grassmann-Cayley algebra and conformal geometric algebra. Basic properties of this framework are discussed, with examples illustrating the advantages in algebraic description and computation of geometric problems.

In classical invariant theory, the guideline of computing is to transform a monomial into a "more standard" form which is usually a polynomial. This often incurs middle expression swell. In Cayley bracket algebra and Clifford bracket algebra, we employ the idea of localized computing, and control the size of the expressions at every minimum step [8]. We do not have enough space left in this paper to further explore this idea and its magic effect in simplifying invariant computation, so we confer to [8] for reference.

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