A Problem of Wetzel's: Fitting Triangle in a Rectangle

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Abstract: A solution to a problem from elementary geometry is presented: a set of polynomial inequalities describing a necessary and sufficient condition for a triangle to fit into a rectangle (given in terms of side lengths). First the possible configurations of the largest similar triangles are classified, then for each of the three classes necessary conditions for the largest similar triangle to fit into the rectangle are derived and shown to be sufficient and finally the obtained conditions are combined. The initial inequalities contain a parameter and (in one case) two additional variables which are eliminated to obtain the final result.

1. Introduction

In 1956, L. Ford asked for a necessary and sufficient condition for a $p \times q$ rug to fit into an $a \times b$ floor. The first answer to this question was given by W. Carver [1]: a $p \times q$ $(p \geq q)$ rectangle fits into an $a \times b$ $(a \geq b)$ rectangle if and only if

$$(p \leq a \wedge q \leq b) \vee (p > a \wedge b \geq \frac{2pqa + (p^2 - q^2)\sqrt{p^2 + q^2 - a^2}}{p^2 + q^2}).$$

In 2000, J. E. Wetzel [3] gave another solution to this problem. The condition he obtained is in the following symmetric form:

$$(p \leq a \wedge q \leq b) \vee (p > a \wedge \left(\frac{a+b}{p+q}\right)^2 + \left(\frac{a-b}{p-q}\right)^2 \geq 2).$$

Similar problem for one triangle to fit into another was solved by K. A. Post [2] in 1993, where he gave a set of 18 inequalities and proved that if one of the inequalities is correct, the first triangle fits in the second; and if the first triangle fits in the second, then at least one of the inequalities is correct.

In article [3], J. E. Wetzel asked to find the necessary and sufficient conditions on sides for the following questions:

(1) a triangle with sides a, b, c to fit in a square with side s;

(2) a square with side s to fit in a triangle with sides a, b, c;

(3) a triangle with sides a, b, c to fit in a rectangle with sides p, q;

(4) a rectangle with sides p, q to fit in a triangle with sides a, b, c.

It is obvious that (1), (2) are special cases of (3), (4) respectively. In this paper, we present a solution to Wetzel's third problem. Our result is:

Theorem 1: The necessary and sufficient condition for a triangle with sides a, b, c to fit into a rectangle with sides p, q is the disjunctive normal form

$$\phi_1(a,b,c,p,q,S) \vee \phi_2(a,b,c,p,q,S) \vee \phi_3(a,b,c,p,q)$$

up to certain permutations of a, b, c and p, q, where S is the area of the triangle with sides a, b, c, that is,

$$S = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)},$$

and ϕ_1, ϕ_2, ϕ_3 are defined by:

$$\begin{aligned}\phi_1 &= 4b^2p^2 + 4c^2q^2 - 16pqS \geq (b^2 + c^2 - a^2)^2 \\ &\quad \wedge (a^2 + b^2 - c^2)p < 4qS < 2b^2 p \\ &\quad \wedge (c^2 + a^2 - b^2)q < 4pS < 2c^2 q, \\ \phi_2 &= a^2 q \leq 2pS \leq apq \wedge a^2 + c^2 \geq b^2 \wedge a^2 + b^2 \geq c^2, \\ \phi_3 &= a^2 \leq p^2 + q^2 \wedge \tan B \leq \frac{p}{q} \wedge \tan C \leq \frac{q}{p},\end{aligned}$$

with respectively.

Our idea to solve this problem is: In the first step to find some geometric properties for the configuration formed by the largest similar triangle that lies in the target rectangle through analyzing the possible rotations and translations of the triangle inside rectangle, then in the second step to establish the equations and inequalities that describe maximal configuration, and finally to get the necessary and sufficient condition through eliminating the parameters (variables other than a,b,c and p,q) in the semi-algebraic systems.

The paper is organized as follows. In Section 2 we will prove that any optimal solution to Wetzel's problem must be one of the three specified configurations, namely, configurations formed by a triangle and a rectangle of Pattern 1, Patter 2 or Patter 3 as defined in this section. In Section 3 we prove that an optimal configuration of Pattern 2 or Pattern 3 satisfies the conjunctive form ϕ_2 or ϕ_3 with respectively. In Section 4 we show that an optimal configuration of Pattern 1 satisfies the conjunctive form ϕ_1. In the last section we state the final result.

2. The Largest Similar Triangle Contained in a Rectangle

The following lemma gives a necessary condition for the largest triangle which is similar to the given triangle and fits into the given rectangle.

Lemma 1: If a triangle ABC fits into a rectangle $KLMN$, the largest similar triangle $A_0B_0C_0$ of ABC that can fit into $KLMN$ satisfies one of the following three properties:

(P1): $A_0B_0C_0$ and $KLMN$ have a common vertex, the other two vertices of the triangle lie respectively in the interior of the two sides of the rectangle that are not adjacent to the common vertex, as shown in Figure 1a;

(P2): One vertex of $A_0B_0C_0$ lies in the interior of a side of $KLMN$, the other two vertices of the triangle lie on the opposite side of the rectangle, as shown in Figure 1b;

(P3): Two vertices of $A_0B_0C_0$ lie on the diagonal vertices of $KLMN$, the third vertex of the triangle lies on the side or in the interior of the rectangle, as shown in the Figure 1c.

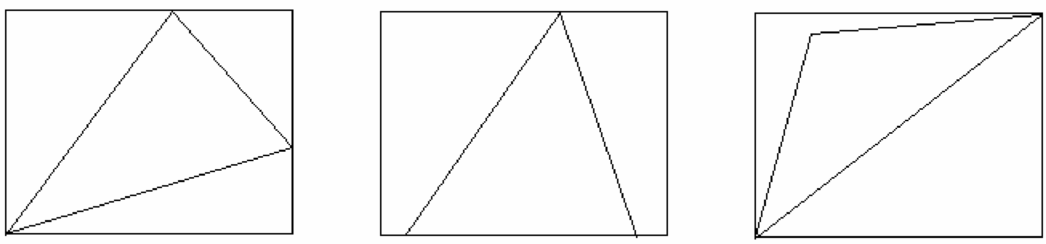

Figure 1a Figure 1b Figure 1c

Proof: The following properties can be easily observed.

(Q1): If a triangle ABC fits into a rectangle $KLMN$ such that there are two adjacent sides of the rectangle, say, MN and NK, containing no vertex of ABC, as shown in figure 2a, then ABC can fit into $KLMN$ so that the four sides of the rectangle contain no vertex of the triangle. This can be done by translating the rectangle as in figure 2b.

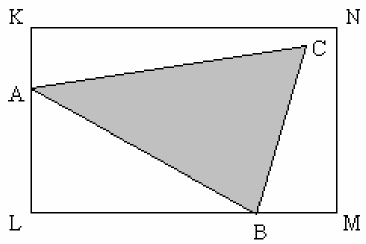

Figure 2a

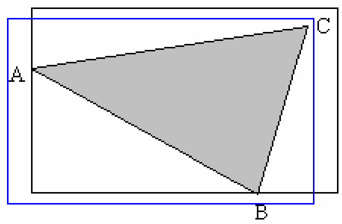

Figure 2b

(Q2): If a triangle ABC fits into a rectangle $KLMN$ such that there are two opposite sides of the rectangle, say, KL and MN, containing no vertex of ABC, and each of the other two sides contains one vertex of ABC, say B in the interior of LM and C in the interior of NK, as shown in figure 3a, then ABC can fit into $KLMN$ so that the four sides of the rectangle contain no vertex of the triangle. This can be done by rotating the triangle ABC as in figures 3b-1, 3b-2 and then translating the rectangle as in figure 3c.

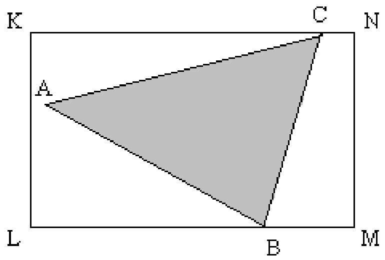

Figure 3a

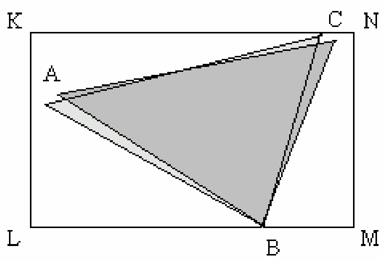

Figure 3b-1

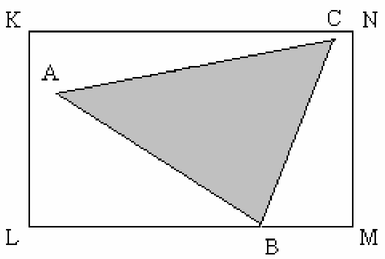

Figure 3b-2

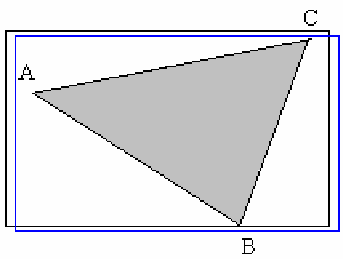

Figure 3c

Let $T(a,b,c)$ be the triangle with sides a,b,c, $R(p,q)$ be the rectangle with sides p,q. Let $\Lambda = \{\lambda \mid \lambda \geq 1, T(\lambda a, \lambda b, \lambda c) \subseteq R(p,q)\}$. If $T(a,b,c)$ fits into $R(p,q)$, then Λ is a non-empty and bounded subset of real numbers. According to the Bolzano-Weierstrass theorem, Λ is a closed set, i.e., $\lambda_0 = \max \Lambda$ exists and $A_0B_0C_0 = T(\lambda_0 a, \lambda_0 b, \lambda_0 c)$ does also fit into the rectangle $R(p,q)$. Let $KLMN$ be the rectangle $R(p,q)$. Consider the configuration formed by $A_0B_0C_0$ and $KLMN$. It is clear from (Q1) and (Q2) that if there are only two vertices of the triangle containing on the sides of the rectangle, then they are either a pair of diagonal vertices of the rectangle as in Figure 1c, or one of them lies on a vertex of the rectangle and another lies on a side of the rectangle which is not adjacent to the common vertex, shown as in Figures 4a and 4b.

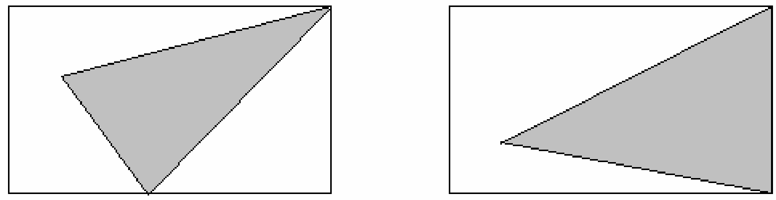

Figure 4a Figure 4b

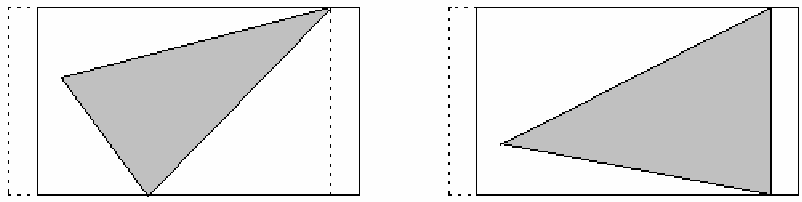

Figure 4c Figure 4d

But in both of the latter two cases, we can translate the rectangle, shown in Figures 4c and 4d, so that $A_0B_0C_0$ fits into the new rectangle, two vertices of $A_0B_0C_0$ lie in the interior of a pair of opposite sides of the rectangle and the third vertex is in the interior of the rectangle as in Figure 2a. According to (Q2) $A_0B_0C_0$ in not the largest similar triangle to fit into $KLMN$. This proves that if the largest similar triangle to fit into the rectangle has only two vertices containing on the sides of the rectangle, then the configuration must satisfy (P3).

If all three vertices of the triangle lie on the sides of the rectangle, then the maximal configurations can be classified into 9 cases according to the number of the common vertices of the triangle and the rectangle, shown as in Figures 5a to 5i.

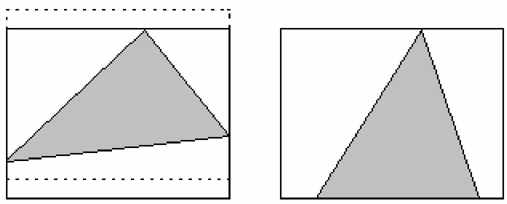

Figure 5a Figure 5b

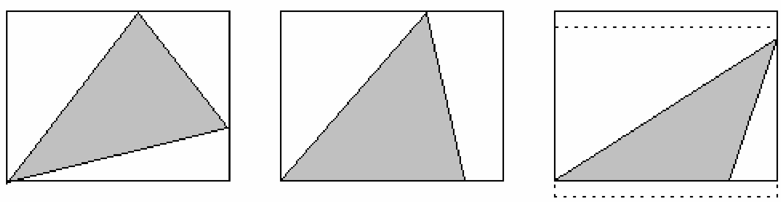

Figure 5c Figure 5d Figure 5e

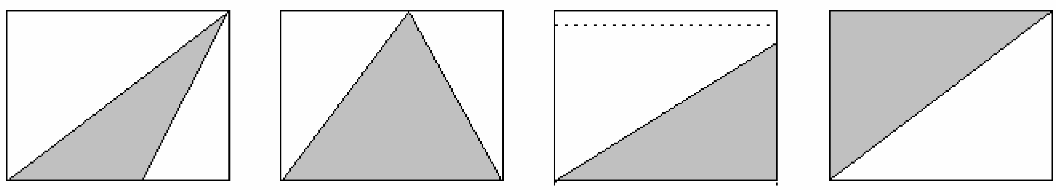

Figure 5f Figure 5g Figure 5h Figure 5i

In Figures 5a and 5b, the triangle and the rectangle have no common vertex, and there is only one common vertex in Figures 5c to 5e, two common vertices in Figures 5f to 5h, three in Figure 5i. It is easy to see that in Figure 5a and 5e we can translate the rectangle as shown in the figures so that two vertices of the triangle lie in the interior of a pair of opposite sides of the rectangle, the third vertex of the triangle lies in the interior of the rectangle, and therefore these two configurations can not be optimal according to (Q2). Among other 7 cases, Figure 5c satisfies (P1); Figures 5b, 5d, 5g and 5h satisfy (P2), Figures 5f and 5i satisfy (P3).

This completes the proof of Lemma 1.

For simplicity of the discussion below, we give the following definition to the configurations of triangle and rectangle.

Definition 1: A configuration formed by a triangle $A_0B_0C_0$ and a rectangle $KLMN$ is called to be of Pattern 1 (or Pattern 2, Pattern 3, respectively), if the $A_0B_0C_0$ and $KLMN$ satisfies the Property (P1) (or (P2), (P3), respectively).

3. Optimal Fitting Configurations of Pattern 2 and Pattern 3

In this section we figure out the optimal fitting configurations of Pattern 2 and 3. First we establish the system of equations and inequalities for each pattern.

Let's start with the configuration formed by the largest similar triangle $A_0B_0C_0$ and the rectangle $KLMN$ of Pattern 3. Without loss of generality, we may assume that $B_0 = L, C_0 = N, A_0 \in KLN$. Then we have

$$p^2 + q^2 = \lambda_0{}^2 a^2 \ge a^2, \quad \tan B = \tan B_0 \le \frac{p}{q}, \quad \tan C = \tan C_0 \le \frac{q}{p}, \quad \lambda_0 \ge 1. \qquad (1)$$

Since

$$\tan B = \frac{4S}{c^2 + a^2 - b^2}, \quad \tan C = \frac{4S}{a^2 + b^2 - c^2},$$

where S is the area of the triangle with sides a, b, c, i.e.,

$$S = \frac{1}{4}\sqrt{(a+b+c)\cdot(a+b-c)\cdot(a-b+c)\cdot(-a+b+c)},$$

the system (1) can be transformed to a semi-algebraic system in variables $a, b, c; p, q$ and parameter λ_0. The next lemma shows that the equations and inequalities in this system are enough for describing a fitting configuration of Pattern 3.

Lemma 2. Let a, b, c be the sides of a triangle ABC, p, q the sides of a rectangle $KLMN$, $p = LM, q = KL$. If the following inequalities

$$a^2 \le p^2 + q^2, \quad \tan B \le \frac{p}{q}, \quad \tan C \le \frac{q}{p}$$

hold, then there exists a point A_0 in the rectangle $KLMN$ such that the triangle $A_0B_0C_0$ with $B_0 = L$, $C_0 = N$ is similar to ABC, and

$$\frac{B_0C_0}{BC} = \frac{C_0A_0}{CA} = \frac{A_0B_0}{AB} \ge 1.$$

Proof: Let $B_0 = L, C_0 = N$. Construct $\angle A_0B_0C_0 = \angle ABC, \angle B_0C_0A_0 = \angle BCA$ so that A_0 is in the same of line LN with K. Then $A_0B_0C_0$ is similar to ABC and A_0 lies in triangle KLN since $\tan B \le p/q$ and $\tan C \le q/p$. It is clear that

$$\frac{B_0C_0}{BC} = \frac{C_0A_0}{CA} = \frac{A_0B_0}{AB} = \frac{\sqrt{p^2+q^2}}{a} \geq 1.$$

Lemma 2 is proved.

For Pattern 2, we may assume $B_0, C_0 \in LM$, $A_0 \in \text{interior}(KN)$ as in Figure 6a. It is easy to observe the following necessary condition:

$$\lambda_0 h_a = q, \quad p \geq \lambda_0 a, \quad \lambda_0 \geq 1, \tag{2}$$

where h_a is the height of the triangle ABC with respect to side BC, and S the area of ABC. Since $h_a = S/(2a)$, the system (2) actually determines a semi-algebraic set of a, b, c, p, q, λ_0. Figures 6b to 6d show that other necessary conditions are needed to guarantee the configuration formed by $A_0B_0C_0$ and $KLMN$ to be of Pattern 2. This additional conditions required are

$$a^2 + c^2 \geq b^2, \quad a^2 + b^2 \geq c^2. \tag{3}$$

To see this, let's assume that $\angle A_0B_0C_0$ or $\angle B_0C_0A_0$ is obtuse, say, $\angle B_0C_0A_0 > 90°$ for example, as in Figure 6b, then we can translate the rectangle $KLMN$ so that $N = A_0$ and $B_0, C_0 \in \text{interior}(LM)$, then rotate the triangle $A_0B_0C_0$ around vertex A_0 so that B_0, C_0 lie in the interior of $KLMN$ as in Figure 6c, and finally translate the rectangle so that all vertices of the triangle lie in the interior of the rectangle, which contradicts to that $A_0B_0C_0$ is the maximal similar triangle of ABC to fit into $KLMN$.

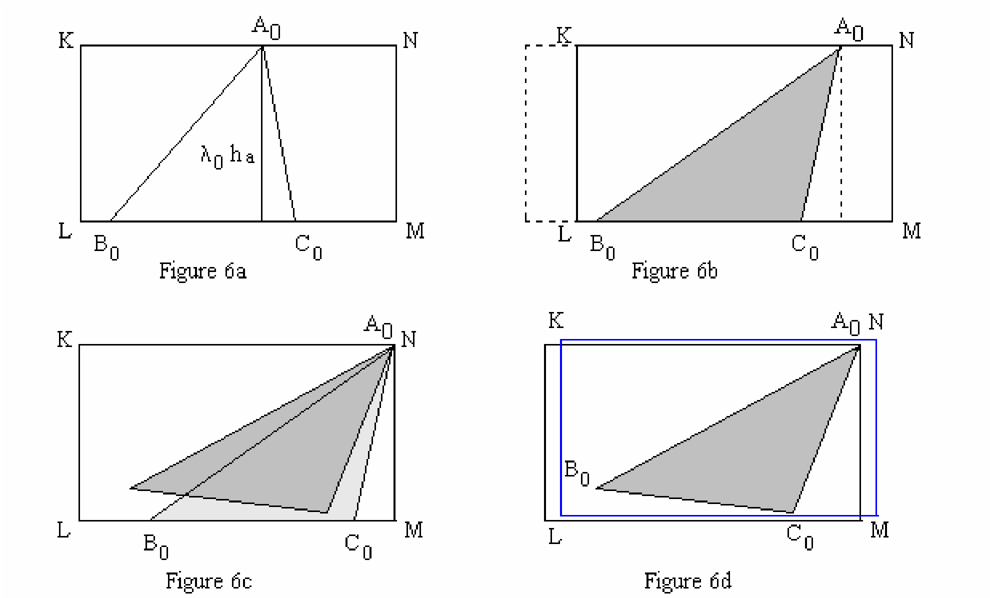

Figure 6a Figure 6b

Figure 6c Figure 6d

Now we prove that these inequalities are also sufficient for the configuration to be of Pattern 2 in essential. Namely, we have the following result.

Lemma 3. Let a, b, c be the sides of a triangle ABC, h_a the height of the triangle ABC with respect to side BC, p, q be the sides of a rectangle $KLMN$, $p = LM, q = KL$. If a, b, c, p, q, λ_0 satisfy the following inequalities

$$\lambda_0 h_a = q, \quad p \geq \lambda_0 a, \quad \lambda_0 \geq 1, \quad a^2 + c^2 \geq b^2, \quad a^2 + b^2 \geq c^2$$

then there exist $A_0 \in NK$, $B_0, C_0 \in LM$ such that the triangle $A_0B_0C_0$ is similar to ABC, and

$$\frac{B_0C_0}{BC} = \frac{C_0A_0}{CA} = \frac{A_0B_0}{AB} \geq 1.$$

Proof: Take $B_0 = L$. Since $LM = p \geq \lambda_0 a$, we can take a point C_0 in LM such that $B_0C_0 = \lambda_0 a$. Construct $\angle A_0B_0C_0 = \angle ABC, \angle B_0C_0A_0 = \angle BCA$ such that point A_0 lies in the same side of line LM with N, K. Then the triangle $A_0B_0C_0$ is similar to ABC, $B_0C_0 / BC = \lambda_0 \geq 1$. According to $a^2 + c^2 \geq b^2, a^2 + b^2 \geq c^2$, we know that $\angle A_0B_0C_0 \leq 90°, \angle B_0C_0A_0 \leq 90°$. This means that A_0 is contained in the strip determined by straight line LK and MN. Since the distance from A_0 to B_0C_0 is $\lambda_0 \cdot h_a = q,$ we know that point A_0 is on NK as claimed. Lemma 3 is proved.

Note that the configurations formed by $A_0B_0C_0$ and $KLMN$ in Lemma 3 are of Pattern 2 ($A_0 \in \text{interior}(NK)$, $B_0, C_0 \in LM$) in general case. The only exception occurs if

$$p = \lambda_0 a \wedge (a^2 + c^2 = b^2 \vee a^2 + b^2 = c^2),$$

and in this case, $A_0 = N \vee A_0 = K$, $B_0 = L, C_0 = M$, the configuration is of Pattern 3.

Now we can give the necessary and sufficient condition for a triangle with sides a, b, c to fit into a rectangle with sides p, q in the way of Pattern 2 or Pattern 3. This can be obtained by eliminating the parameter λ_0 in systems (1), (2) and (3). We have the following result.

Theorem 2. Let ABC be the triangle with sides a, b, c, $KLMN$ the rectangle with sides p, q, $A_0B_0C_0$ the largest triangle similar to ABC that can fit into $KLMN$. Then ABC can fit into $KLMN$ and the configuration formed by $A_0B_0C_0$ and $KLMN$ is of Pattern 2 or Pattern 3, if and only if either

$$(a^2 q \leq 2pS \leq apq \wedge a^2 + c^2 \geq b^2 \wedge a^2 + b^2 \geq c^2)$$
$$\vee (a^2 \leq p^2 + q^2 \wedge 4pS \leq (a^2 + b^2 - c^2)q \wedge 4qS \leq (c^2 + a^2 - b^2)p),$$

up to permutations of a, b, c and p, q, where S is the area of triangle ABC.

Proof: Let $S(a, b, c)$ be the area of the triangle with sides a, b, c, and

$$F = \tan B - \frac{p}{q} = \frac{4S(a,b,c)}{c^2 + a^2 - b^2} - \frac{p}{q}, \quad G = \tan C - \frac{p}{q} = \frac{4S(a,b,c)}{a^2 + b^2 - c^2} - \frac{q}{p}.$$

Then it is easy to see that the semi-algebraic set defined by

$$p^2 + q^2 = \lambda_0^2 a^2 \wedge F \leq 0 \wedge G \leq 0 \wedge \lambda_0 \geq 1$$

can also be generated by formula

$$(p^2 + q^2 \geq a^2 \wedge F \leq 0 \wedge G \leq 0) \text{ and } \lambda_0 = \frac{\sqrt{p^2 + q^2}}{a},$$

while the parameter λ_0 has been eliminated in the first operand

$$p^2 + q^2 \geq a^2 \wedge F \leq 0 \wedge G \leq 0,$$

that is,

$$a^2 \leq p^2 + q^2 \wedge 4pS \leq (a^2 + b^2 - c^2)q \wedge 4qS \leq (c^2 + a^2 - b^2)p.$$

This gives a necessary and sufficient condition for the triangle $T(a, b, c)$ to fit into the rectangle $R(p, q)$, and the configuration formed by $R(p, q)$ and the largest triangle similar to $T(a, b, c)$ that can also fit in $R(p, q)$ is of Pattern, up to a permutation of a, b, c and p, q.

Similarly, the semi-algebraic set determined by (2) together with (3) can be represented by

$$(q \ge h_a \wedge p \ge \frac{a}{h_a} \cdot q \wedge c^2 + a^2 \ge b^2 \wedge a^2 + b^2 \ge c^2) \text{ and } \lambda_0 = \frac{q}{h_a},$$

and the parameter λ_0 has been eliminated in the first operand, too. Substituting $h_a = S(a,b,c)/(2a)$ into this formula we obtain

$$a^2 q \le 2pS \le apq \wedge c^2 + a^2 \ge b^2 \wedge a^2 + b^2 \ge c^2.$$

Theorem 2 is proved.

4. Optimal Fitting Configuration of Pattern 1

For configuration of Pattern 1, we may find some necessary conditions through analyzing the geometric properties. Let $A_0B_0C_0$ be a triangle inscribed in the rectangle $KLMN$ such that $A_0 = L$, B_0 in the interior of MN and C_0 in the interior of KN as shown in the Figure 7. Take point D in KL such that B_0D is parallel to LM, and point F in A_0B_0 such that C_0F is perpendicular to A_0B_0. Let E be the intersection point of A_0C_0 and B_0D, G the intersection of C_0F and LM. Then the following inequalities hold:

$$\angle A_0 < 90°,$$

$$\angle C_0 > \angle B_0NE > \angle MNL = \arctan\frac{p}{q}, \quad \angle B_0 > \angle LNK = \arctan\frac{q}{p}, \tag{4}$$

$$\frac{p}{q} = \frac{A_0B_0}{C_0G} < \frac{A_0B_0}{C_0F} = \frac{c}{h_c}, \quad \frac{q}{p} < \frac{b}{h_b}.$$

These inequalities are not sufficient since if triangle $A_0B_0C_0$ satisfies (4) then any triangle $A_1B_1C_1$ similar to $A_0B_0C_0$ also satisfy the inequalities.

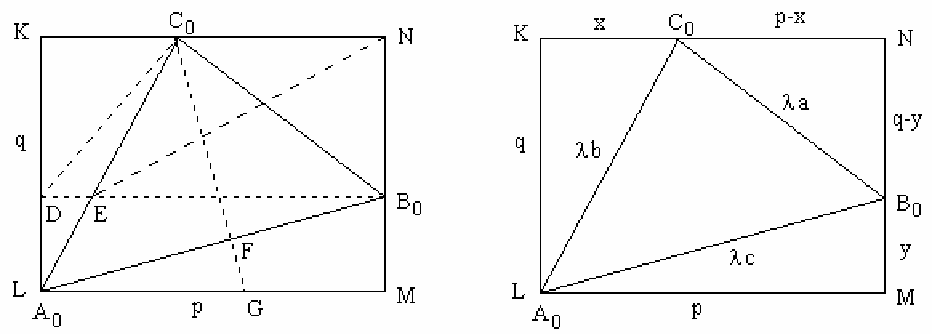

Figure 7 Figure 8

We can construct the necessary and sufficient condition by establishing the semi-algebraic system of the configuration and then eliminating the parameter for Pattern 1 as following way. We may assume $KC_0 = x, MB_0 = y$ as in Figure 8. Then any configuration of Pattern 1 satisfies the following equations

$$\begin{cases} x^2 + q^2 = \lambda_0{}^2 b^2, \\ y^2 + p^2 = \lambda_0{}^2 c^2, \\ (p-x)^2 + (q-y)^2 = \lambda_0{}^2 a^2, \end{cases} \tag{5}$$

and inequalities

$$\lambda_0 \ge 1,\ \ 0 < x < p,\ \ 0 < y < q. \qquad (6)$$

On the other hand, any (real positive) solution of the semi-algebraic set determined by (5) and (6) also generates a configuration of Pattern 1, that is, if there exist x, y, λ_0 satisfying (5) and (6), then $A_0B_0C_0$ can be moved into the rectangle $KLMN$ so that the configuration is of Pattern 1. We have the following result.

Lemma 4. Let a, b, c be the sides of a triangle, S the area of this triangle, $p > 0, q > 0$. Then the equation system (5) has real solution λ_0, x, y satisfying inequalities (6) if and only if

$$4b^2p^2 + 4c^2q^2 - 16pqS \ge (b^2 + c^2 - a^2)^2,$$
$$(a^2 + b^2 - c^2)p < 4qS < 2b^2p,$$
$$(c^2 + a^2 - b^2)q < 4pS < 2c^2q.$$

Proof: First we transform the equation system (5) into triangular form. Using pseudo-remainder or resultant provided in Maple, the system (5) can be transformed to

$$\begin{cases} (b^2+c^2-a^2)^4\lambda_0{}^4 - 8(b^2p^2+c^2q^2)\cdot(b^2+c^2-a^2)^2\lambda_0{}^2 \\ \qquad\qquad + 16(b^2p^2+c^2q^2)^2 - 256p^2q^2S^2 = 0, \\ 4(b^2+c^2-a^2)\cdot p\cdot x - (b^2+c^2-a^2)^2\lambda_0{}^2 - 4(b^2p^2-c^2q^2) = 0, \\ 4(b^2+c^2-a^2)\cdot q\cdot y - (b^2+c^2-a^2)^2\lambda_0{}^2 + 4(b^2p^2-c^2q^2) = 0, \end{cases} \qquad (7)$$

under condition $b^2 + c^2 - a^2 \ne 0$, and to the following trivial form

$$x = 0,\ y = 0,\ q = \lambda_0 b,\ p = \lambda_0 c, \qquad (8)$$

if $b^2 + c^2 - a^2 = 0$. Since in latter case, we have $S = 1/2\,bc$, (8) is also satisfying equation system (7). Thus the semi-algebraic set defined by (5) and (6) equals to the semi-algebraic set defined by (7) and (5). It is easily observed that equation system (7) can be decomposed to the following two branches:

$$E_1:\ \begin{cases} \lambda_0{}^2 = \dfrac{4b^2p^2 + 4c^2q^2 + 16pqS}{(b^2+c^2-a^2)^2}, \\ x = \dfrac{2b^2p + 4qS}{b^2+c^2-a^2}, \\ y = \dfrac{2c^2q + 4pS}{b^2+c^2-a^2}, \end{cases} \qquad E_2:\ \begin{cases} \lambda_0{}^2 = \dfrac{4b^2p^2 + 4c^2q^2 - 16pqS}{(b^2+c^2-a^2)^2}, \\ x = \dfrac{2b^2p - 4qS}{b^2+c^2-a^2}, \\ y = \dfrac{2c^2q - 4pS}{b^2+c^2-a^2}, \end{cases}$$

Now we prove that λ_0, x, y in branch E_1 do not satisfy inequality system (6). From $x > 0, y > 0$, we know $b^2 + c^2 - a^2 > 0$, and immediately obtain

$$2b^2p + 4qS < (b^2 + c^2 - a^2)p, \quad 2c^2q + 4pS < (b^2 + c^2 - a^2)q,$$

and therefore,

$$4qS < (-a^2 - b^2 + c^2)p, \quad 4pS < (-c^2 - a^2 + b^2)q,$$

which leads to $S < 0$ since at least one of $\angle B$ and $\angle C$ is acute. This is impossible. So we proved that positive real solution λ_0, x, y of system (5) and (6) is given by E_2. Thus, the condition for the existence of positive real solution λ_0, x, y of the original system in terms a, b, c and p, q is:

$$\frac{4b^2p^2 + 4c^2q^2 - 16pqS}{(b^2+c^2-a^2)^2} \ge 1,\ \ 0 < \frac{2b^2p - 4qS}{b^2+c^2-a^2} < p,\ \ 0 < \frac{2c^2q - 4pS}{b^2+c^2-a^2} < q.$$

To simplify this condition, we show we show $b^2 + c^2 - a^2 > 0$. Otherwise,

$$2b^2p - 4qS < 0, \quad 2c^2q - 4pS < 0,$$

and therefore,

$$4b^2c^2pq < 16pqS^2 = 4pqb^2c^2\sin^2 A,$$

which leads to $\sin A > 1$. This finishes the proof of Lemma 4.

The above discussion on Pattern 1 can be summarized to the following theorem.

Theorem 3. Let ABC be a triangle with sides a,b,c, $KLMN$ be a rectangle with sides p,q, $A_0B_0C_0$ the largest triangle similar to ABC that can fit into $KLMN$. Then if and only if

$$4b^2p^2 + 4c^2q^2 - 16pqS \geq (b^2+c^2-a^2)^2,$$
$$(a^2+b^2-c^2)p < 4qS < 2b^2p,$$
$$(c^2+a^2-b^2)q < 4pS < 2c^2q.$$

up to certain permutation of a,b,c and p,q, ABC can fit into $KLMN$ and the configuration formed by $A_0B_0C_0$ and $KLMN$ is of pattern 1.

To conclude this section we show the geometric meaning of inequalities in Theorem 3. Note that by substituting the following formulas

$$a^2+b^2-c^2 = 2ab\cos C, \quad S = \frac{1}{2}ab\sin C = \frac{1}{2}bh_b,$$
$$c^2+a^2-b^2 = 2ac\cos B, \quad S = \frac{1}{2}ac\sin B = \frac{1}{2}ch_c,$$

in the latter two forms, we get

$$\frac{1}{\tan C} < \frac{q}{p} < \frac{b}{h_b}, \quad \frac{1}{\tan B} < \frac{p}{q} < \frac{c}{h_c},$$

and therefore,

$$\angle C > \arctan\frac{p}{q}, \quad \angle B > \arctan\frac{q}{p},$$

as we have observed from Figure 7.

5. Conclusion

Lemma 1 shows that largest triangles that can fit into a rectangle satisfies one of the (P1), (P2) or (P3), Theorem 2 proves that configuration corresponding to (P2) or (P3) satisfies form $\phi_2 \vee \phi_3$, and Theorem 3 proves that configuration corresponding to (P1) satisfies ϕ_1. This leads to a necessary and sufficient for a triangle of sides a,b,c to fit into a rectangle with sides p,q, in terms of some polynomial inequalities on of a,b,c and p,q, as stated in the Theorem 1.

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