

# Conic Section and Eccentricity: How The Geometer's Sketchpad (GSP) Enhance Students' Understanding

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## Abstract

This paper aims to describe how The Geometer's Sketchpad (GSP) enhances students' understanding on the topic of a conic section and eccentricity. A conic section is the locus of a point having a distance from a fixed point that is in constant ratio to its distance from a fixed line. The fixed point is called the focus, and the fixed line is called the directrix. The constant ratio is called the eccentricity of the conic, and its value gives the kind of conic section. A circle has eccentricity 0, an ellipse between 0 and 1, a parabola is 1 and hyperbolae have eccentricity greater than 1.

GSP is one of the dynamic mathematics software that provides opportunities for teachers and students to use their ability to access, drag, visualize, and create graphical representation of conics section and the eccentricity. GSP enable students to construct graphical and numerical representation together. These enhance students' understanding in interpreting eccentricity and conics section both visually and numerically.

## Introduction

This paper aim to describe how The Geometer's Sketchpad (GSP) enhances students' understanding on the topic of eccentricity and conic section. Conic section deals with symbols and notation, abstract concepts formulas, and proofs. The abstract concept of the *eccentricity* of conic section is one of the topics that students learned without understanding. That is because the students have to work on this topic through drill, practice and memorize the formulas without understanding. By using GSP the students are able to develop their representation in mathematics including grasp basic concepts before they study advanced conic section in analytic geometry.

## Students' Understanding in Mathematics

According to Sierpinska (1994), understanding is the mental experience of a person by which he/she relates an object to another object. Personal understanding of a concept is grasping or acquiring the meaning of the object. Understanding is a matter of being able to do a variety of thought-provoking things with a topic, such as explaining, finding evidence and examples generalizing, applying and representing the topic in new ways. (Blythe,1998).

Richard Skemp (1978) made a powerful statement about mathematical understanding. He described two different meanings generally associated with "understanding" in mathematics. They are *relational understanding* and *instrumental understanding*. He explained the first one as "knowing both what to do and why" where as the instrumental understanding he described as "rules

without reasons.” According to Skemp, *relational understanding* involves understanding structures and connections within concepts. Relational knowledge of mathematics is characterized by the possession of conceptual structures that enable the student to construct several plans for performing a given task. In contrast, *instrumental understanding* demonstrates ability to manipulate formulas and carry out operation. The instrumental knowledge of mathematics is knowledge of a set of “fixed plans” for performing mathematical tasks. The characteristic of these plans is that they prescribe a step-by-step procedure to be followed in performing a given task, with each step determining the next. (Skemp, 1978, p.14).

In addition, the van Hiele theory of geometric thought had great impact on the geometric thinking (van Hiele & van Hiele-Geldof, 1958). van Hiele theory consists of five-level hierarchy of ways of understanding of geometry. They are visualization, analysis, informal deduction, deduction and rigor. Each of the five levels describes the thinking processes used in geometric contexts. van Hiele describe that the product of thought at each level are the same as the objects of thought at the next level. The students have to create their ideas at one level so that relationships among these ideas can become the focus of the next level. Moreover Meyer (2001) explains that students bridge the gap between concrete and increasingly abstract levels through their creation and use of models, drawings, diagrams, tables or symbolic notations and technology is a good tool for mathematical modeling.

## **Empower Through Tools: The Geometer’s Sketchpad**

The Geometer’s Sketchpad is one of the dynamic mathematics software that provides opportunities for students to investigate and discover mathematics concepts in particular geometric concepts. GSP empower students to use their ability to create graphical representation, which will enable them to develop their mathematical thinking skills, concepts and understanding. By using GSP students learn through exploring, investigating and discovering.

GSP was introduced in Thailand since the year 2000, in year 2004 GSP was translated into Thai language and used widely in Thailand. More than 3,000 mathematics teachers were trained to use GSP as a tool in their mathematics classes. At least 200 workshops on the use of GSP were conducted by various agency in Thailand such as SEAMEO RECSAM (Southeast Asian Ministry of Education Organization, Regional Centre for Science And Mathematics), Ministry of Education Thailand under World Bank loan project on “*Secondary Education Quality Improvement on Mathematics (SEQI): Module 3*, The Dream School Project of Ministry of Education Thailand, The Institute for the Promotion of Teaching Science and Technology (IPST), Rajabhat Universities, and Chiangmai University.

## **How GSP Enhance Students’ Understanding on Conic Section and Eccentricity**

A conic section can be defined as the locus of a point having a distance from a fixed point that is in constant ratio to its distance from a fixed line. This constant ratio is called the *eccentricity* of the conic; this quantity is denoted by  $e$ . The fixed point is called the *focus*, and the fixed line is called the *directrix*. The eccentricity characterizes the shape of the curve. A circle has eccentricity 0, an ellipse between 0 and 1, a parabola 1, and hyperbola have eccentricity greater than 1.

The following activities describe how GSP can enhance students’ understanding in interpreting eccentricity and conic section both visually and numerically. With GSP students are

able to create graphical representation of eccentricity and conic section. GSP enable students to construct graphical and numerical representation together.

**Activity I:** Investigation the eccentricity  $0 \leq e \leq 1$

The construction of a conic section is determined by given a line and a point not on the line and a constant ratio. We shall construct the locus of a point such that the *ratio* of the distance from a fixed point to the distance from a fixed line is constant and denoted as an eccentricity ( $e$ ).

Sketch and Investigate:

1) Start the Geometer's Sketchpad program and choose *New Sketch* from *File* menu, then construct a right triangle, the ratios and eccentricity as follows:

- construct a right triangle ABC;
- construct point D on AB;
- construct a line through point D, parallel to BC;
- construct point E where the parallel line intersects AC;
- construct segment AD and DE, then hide the parallel line; now we have ABC and ADE are similar triangles (as show in Figure 1),

so that the ratios of segment lengths  $\frac{m\overline{AD}}{m\overline{AB}}$  and

$\frac{m\overline{DE}}{m\overline{BC}}$  are equal;

- measure the segments  $\overline{AD}$ ,  $\overline{DE}$ ,  $\overline{AB}$  and  $\overline{BC}$
- calculate the ratios  $\frac{m\overline{AD}}{m\overline{AB}}$  and  $\frac{m\overline{DE}}{m\overline{BC}}$ ;
- make an action button to animate point D to observe the value of the ratios;
- construct the action buttons to move point  $D$  to point  $A$  and to move point  $D$  to point  $B$ .

2) Construction of a *focus point* and the *directrix line*

- construct a line  $l$  to be the directrix line and point  $F$ , not on line  $l$ . Point  $F$  will be the focus point;
- construct a line through point  $F$  perpendicular to the directrix line and point  $X$  is a movable point on the perpendicular line;
- construct a line parallel to the directrix through point  $X$ .

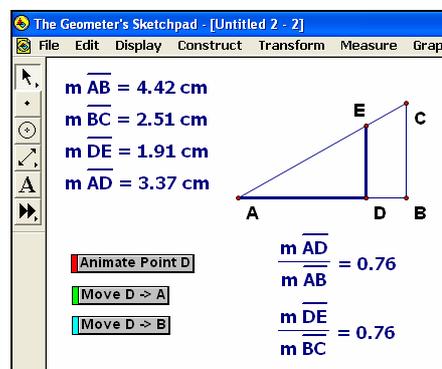


Figure 1: Construction of a right triangle ABC and the ratios

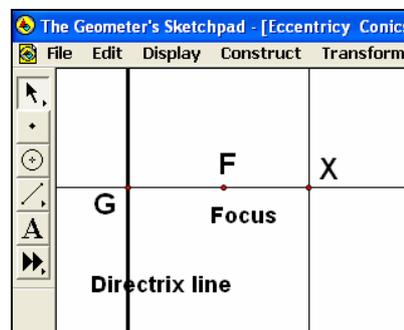


Figure 2: A focus point and the directrix line

3) Construction of eccentricity ( $e$ ) with the marked ratio  $\frac{m\overline{DE}}{m\overline{BC}}$

- mark the ratio of DE to BC by selecting segment  $\overline{DE}$  and  $\overline{BC}$  in that order, then in the *Transform* menu choose *Mark Segment Ratio*;
- select point G and apply *Mark center* from the *Transform* menu;
- select point X and choose *Dilate* from the *Transform* menu, the result is a point X' satisfying

$$\frac{|GX'|}{|GX|} = e = \frac{m\overline{DE}}{m\overline{BC}}$$

- construct segments  $\overline{GX'}$  and  $\overline{GX}$ , then measure and calculate the ratio  $\frac{m\overline{GX'}}{m\overline{GX}}$  ;

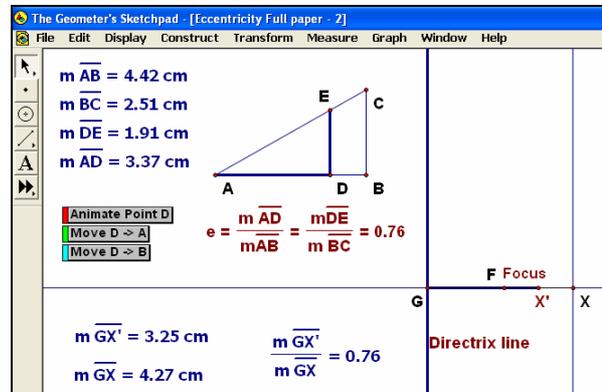


Figure 3: Construction of eccentricity ( $e$ ) with the marked ratio

- drag point X or point D to observe the behavior of point X' and the ratios.

4) Construction of locus of points of a conic section

In order to construct the locus of points in a plane with the constant ratio of the distance from a fixed point to its distance from a fixed line, we shall construct a circle with center  $F$  and radius equal to  $\overline{GX'}$ . Let point  $I$  and point  $J$  are two points of intersection of the circle and the parallel line. Being on a circle, point  $I$  has distance  $|GX'|$  from  $F$ , and being on parallel line point  $I$  has distance

$|GX|$  from the directrix line. Since  $\frac{|GX'|}{|GX|} = e$  ; we have  $\frac{|FI|}{|IK|} = e$  as desired.

Using GSP to construct the locus points of a conic section as follows:

- construct a circle with center  $F$  and radius equal to  $\overline{GX'}$ ;
- point  $I$  and point  $J$  are two points of intersection of the circle and the parallel line;
- construct a line through point  $I$  perpendicular to the directrix line;
- construct point  $K$  where the perpendicular line intersect the directrix line;
- construct segments  $\overline{FI}$  and  $\overline{IK}$ ;
- measure segments  $\overline{FI}$  and  $\overline{IK}$  and calculate the ratio  $\frac{m\overline{FI}}{m\overline{IK}}$  and rename the ratio to

$$\frac{m\overline{FI}}{m\overline{IK}} = \frac{m\overline{GX'}}{m\overline{GX}} = \frac{m\overline{DE}}{m\overline{BC}} = e ;$$

- select point X and point I, then in the *Construct* menu choose *Locus*;
- select point X and point J, then in the *Construct* menu choose *Locus*;
- drag point X along the perpendicular line to drive point  $I$  around ellipse and observe that the value of ratio  $e$  is always constant.

The locus of a point with the constant ratio of the distance from a fixed point to the distance from a fixed line is shown in Figure 4.

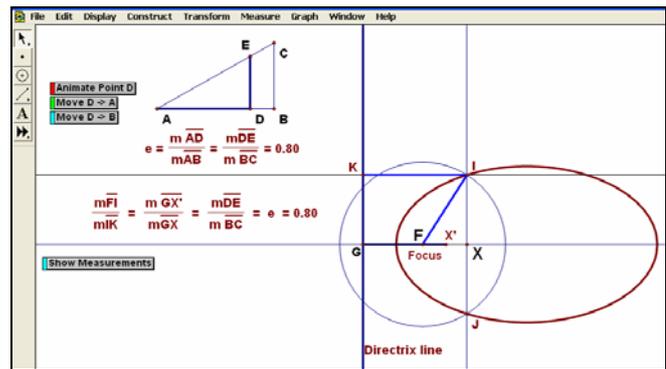


Figure 4: Locus of points of a conic section with ratio  $e$

5) Investigation the eccentricity and the locus of points of a conic section

Investigation 1:  $e$  close to 0

- drag point D near point A, the ratio  $e$  is small and the locus of points is *ellipse* shape. The shape is nearly circular (as shown in Figure 5).

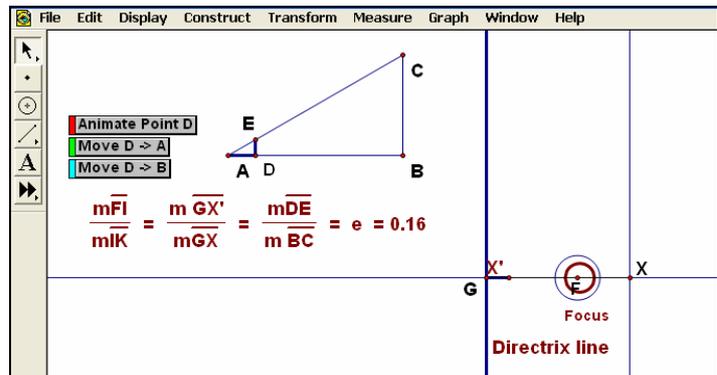


Figure 5: Ellipse is nearly circular when the ratio  $e$  is small.

Investigation 2:  $e$  close to 1

- drag point D near point B, the ratio  $e$  is closed to 1 and the ellipse shape is quite flat (as shown in Figure 6).
- animate point D to observe the behave of the eccentricity and the locus of points.

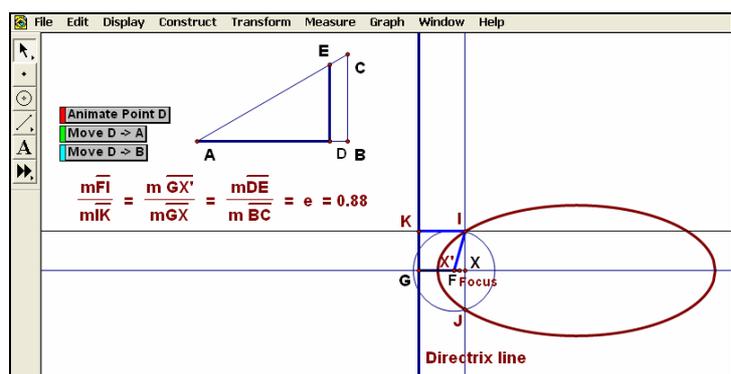


Figure 6: Ellipse is quite flat when the ratio  $e$  is close to 1.

*Investigation 3:  $e = 1$*

- observe the locus of points when  $e = 1$  by dragging point D to point B;
- clicking the action button D -> B, then the locus of point is *parabola* shape (as shown in Figure 7).

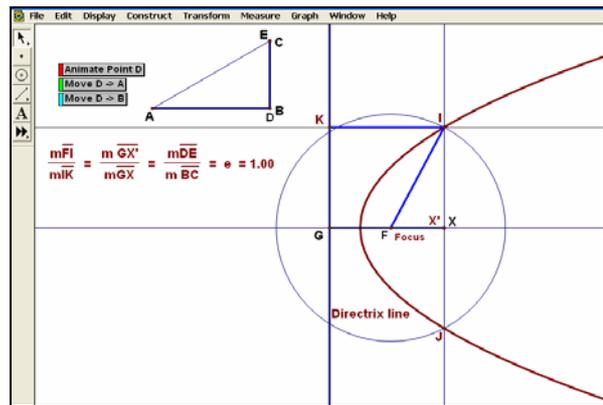


Figure 7: The locus of points is *parabola* when  $e = 1$

From the investigations above, the students are able to understand how eccentricity value can shape the locus of points. They find out that it is an *ellipse* shape when  $0 \leq e \leq 1$  and the locus of points is *parabola* when  $e = 1$ .

**Activity II:** Investigation the eccentricity  $e > 1$

The construction of the locus of points when the eccentricity is greater than 1 is similar to the construction of *Activity 1: Investigation of the eccentricity  $0 \leq e \leq 1$* . We shall begin with the construction of a right triangle, the ratios, the eccentricity, a *focus point* and the *directrix line*.

In Activity 2, we will assign the eccentricity with the marked ratio  $\frac{mBC}{mDE}$ , in order to create the ratio  $e$  exceeds 1.

Sketch and Investigate:

- 1) Construct a right triangle, the ratios, the eccentricity ( $e$ ), the focus point and the directrix line the same as in the *Activity 1: Investigation of the eccentricity  $0 \leq e \leq 1$* ;
- 2) Construction of eccentricity with the marked ratio  $\frac{mBC}{mDE}$ ;
- 3) Construct a *focus point*, the *directrix line* and locus of points of a conic section the same as Step 2, Step 3, and Step 4 of the *Activity 1*. We shall have the locus of points of a conic section with the ratio  $e > 1$  called *hyperbola*.

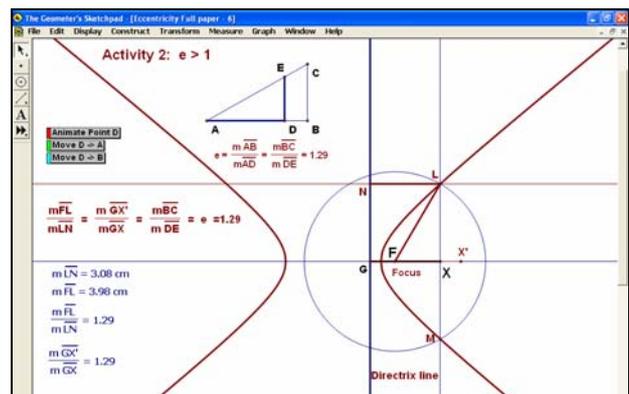


Figure 8: The locus of points is *hyperbola* shape when the ratio  $e > 1$

- 4) Investigation the locus of points when  $e > 1$  by dragging point D, animating point D, clicking the action button D -> A and button D -> B.

**Activity III:** Exploring an ellipse and the eccentricity in a Cartesian plane

In a Cartesian plane, an ellipse has an equation based on the values of  $a$  and  $b$ . Where  $a$  is the semi-major axis and the  $b$  is called the semi-minor axis.

The two foci are on the major axis, and are marked  $F_1$  and  $F_2$ . The distance to the focus from centre  $O$  is  $c$ .

The equation for the ellipse is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  and the eccentricity ( $e$ ) of an ellipse is

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}$$

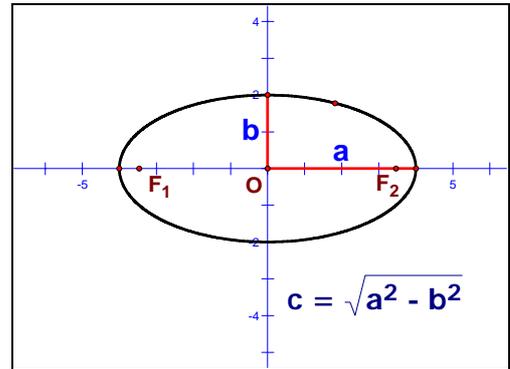


Figure 9 : An ellipse in a Cartesian plane

Sketch and Investigate:

- 1) In a new sketch, choose *New Parameter* from the *Graph* menu, naming it  $a$ . Let  $a = 4$ . Use the same technique to create a parameter  $b$ , let  $b = 2$ .

- 2) Choose *Plot New Function* from the *Graph* menu and graph the function

$$f(x) = \sqrt{b^2 \left(1 - \frac{x^2}{a^2}\right)}$$

$$g(x) = -\sqrt{b^2 \left(1 - \frac{x^2}{a^2}\right)}$$

- 3) Choose *Calculate* from the *Measure* menu to find  $\sqrt{a^2 - b^2}$  and change its label to  $c$ .
- 4) Choose *Calculate* from the *Measure* menu to find the value of eccentricity

$$e = \frac{c}{a}$$

- 5) Explore the eccentricity and ellipse by changing value of the parameters  $a$  and  $b$  manually or clicking at the animation buttons.

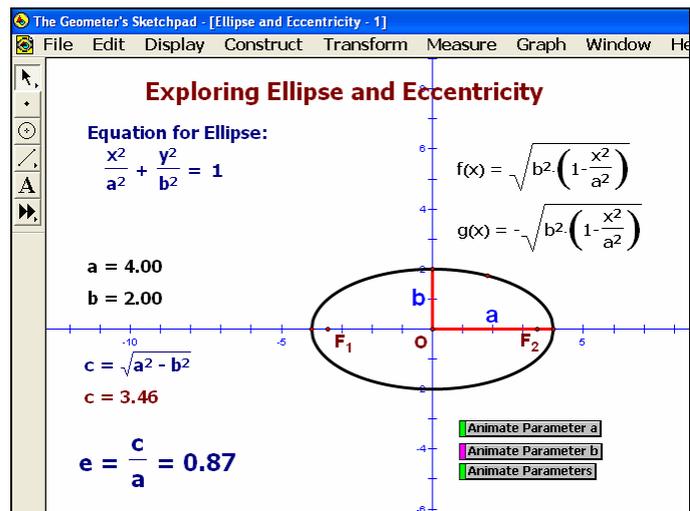


Figure 10: Exploring an ellipse and the eccentricity

**Activity IV:** Exploring a hyperbola and the eccentricity in a Cartesian plane

In a Cartesian plane, the equation for the hyperbola is  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , the asymptotes are

$y = \pm \frac{b}{a}x$  and the eccentricity is  $e = \frac{\sqrt{a^2 + b^2}}{a}$  where  $a$  is the semi-major axis and  $b$  is the semi-minor axis.

Sketch and investigate:

- 1) In a new sketch, choose *New Parameter* from the *Graph* menu, naming it *a*. Let  $a = 5$ . Use the same technique to create a parameter *b*, let  $b = 3$

- 2) Choose *Plot New Function* from the *Graph* menu and graph the function

$$f(x) = \sqrt{b^2 \left( \frac{x^2}{a^2} - 1 \right)}$$
 and

$$g(x) = -f(x).$$

- 3) Choose *Plot New Function* from the *Graph* menu and graph the function

$$h(x) = \frac{b}{a}x \quad \text{and}$$

$$q(x) = -h(x).$$

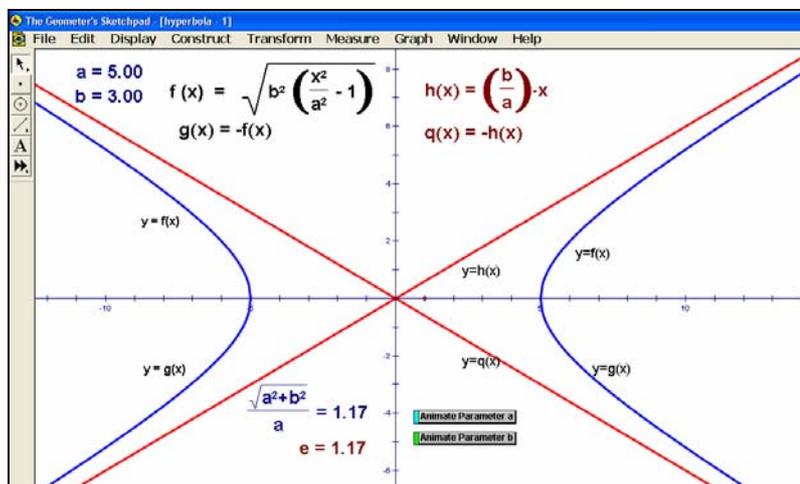


Figure 11: Exploring hyperbola and the eccentricity

- 5) Choose *Calculate* from the *Measure* menu to find  $\frac{\sqrt{a^2 + b^2}}{a}$  and change its label to *e*.
- 6) Explore the eccentricity and hyperbola by changing value of the parameters *a* and *b* manually or clicking at the animation buttons.

From the investigations above, the students are able to understand that the shape of ellipse and hyperbola depend on the value of the eccentricity (*e*). They find out that it is an *ellipse* when  $0 \leq e \leq 1$  and *hyperbola* when  $e > 1$ .

## Conclusions

This paper had shown how GSP can enhance students' understanding in interpreting eccentricity and conics section both visually and numerically. GSP is a very useful and effective tool for both teacher and students. GSP changes the way mathematics is taught. It is clear that, with GSP students are able to create graphical representation of eccentricity and conic section. GSP enable students to construct graphical and numerical representation together.

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