Some Optimization Problems in Two and Three Dimensions

Wei-Chi Yang
e-mail: wyang@radford.edu
Department of Mathematics and Statistics
Radford University
Radford, VA 24142
USA

Abstract

In this paper, we will explore that if we are given two differentiable curves or surfaces in two or three dimensions respectively, how we can find the local maximum or minimum distance between these two curves or two surfaces. We begin with a simpler version from a two dimensional case, which is inspired by the software, ClassPad Manager (see [1]), integrating Dynamic Geometry features with a Computer Algebra System (CAS). We further demonstrate that the geometric signifcance of finding extremum distances between two curves or two surfaces is consistent when we apply Lagrange Multiplier Method. Finally, we show how we can find the minimum distance between two surfaces with the help of Maple 10 (see [4]).

1 Introduction

Finding minimum distance between two surfaces has many applications in robotic engineering (see [2] and [3]). In section 2.1, we describe one key observation of finding the minimum distance inspired by the integration of Dynamic Geometry with a Computer Algebra System (CAS). We next describe methods of finding global maximum or minimum distance in two and three dimensional cases. In this paper, we assume two curves in two dimensions or two surfaces in three dimensions are disjoint unless otherwise specified. For simplicity, we will calculate the square distance $|\mathbf{x}-\mathbf{y}|^2$ instead of $|\mathbf{x}-\mathbf{y}|$, where $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^2$ or $\mathbb{R}^3$.

2 Two Dimensional Case

2.1 A Simple Observation with A Dynamic Geometry and CAS Software

Before we find the local maximum or minimum square distance between two given curves, we note the following observations:

1. Given a point A point (in some neighborhood) on $y = f(x)$, we can attempt to find a corresponding point B on $y = g(x)$ so that the vectorAB is perpendicular to the tangent line at B Therefore, we attempt to solve for c so that the following equation is met
$$\left(\frac{g(c) - f(x)}{c - x}\right) \cdot g'(c) = -1. \tag{1}$$
2. When c can be solved explicitly in terms of x, say $c = G(x)$, then $y = G(x)$ is the solution to this problem.
3. When c can not be explicitly solved, then the solution c will satisfy the following implicit equation:
$$\left(\frac{g(c) - f(x)}{c - x}\right) \cdot g'(c) + 1 = 0. \tag{2}$$

We note that many dynamic geometry software has the capability of constructing geometric figures and tracing the locus after an animation. We will use ClassPad Manager [1] to explore how dynamic animations can help making the observations more accessible.

Example 1 *Let C_1 be the circle represented by $x^2 + (y - 1)^2 = 1$ and C_2 be the parabola represented by $g(x) = -x^2 - 1$. We show that given a point C in some neighborhood on C_1, we can find the corresponding point A on C_2 so that CA is perpendicular to the tangent line at A. For the construction below, we start with a point A on C_2 first and find the corresponding point C on C_1, which reverses what we want to do but this will become clear later. We describe the steps of constructing this animation below:*
Step 1. We construct two curves, C_1 and C_2.
Step 2. We next construct the tangent to a curve at the point A of C_2.
Step 3. We then construct a perpendicular line (normal line) to the tangent line at the point A.
Step 4. We construct the intersection between the normal line and C_1 and call such point C. (We note that due to the limitation of the software, we can only find the intersection geometrically for certain curves.
Step 5. We animate the point A along curve C_2. We show some of the screen shots below:

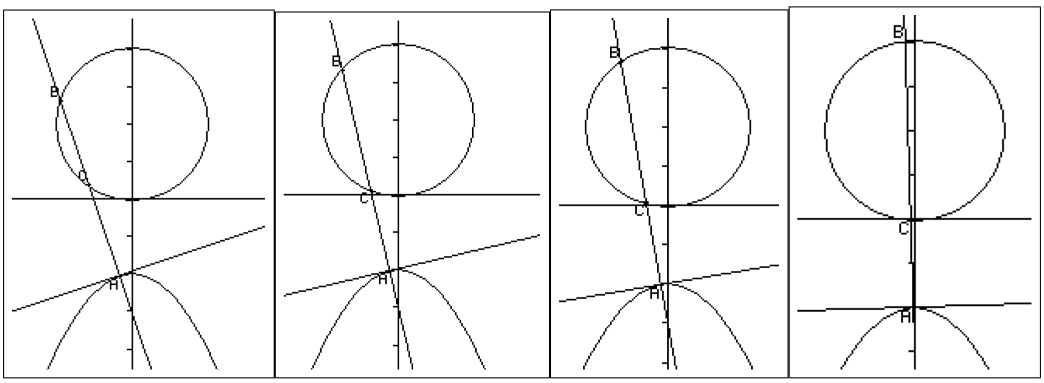

***Figure** 1(a),(b),(c) and (d) Animation from ClassPad*

Step 6. We collect the point C on C_1 first and the corresponding point A on C_2 later after the *animation. We collect only the $x-values$ for points of C and A below. Note that the first column represents the $x-values$ of C and the second column represents the $x-values$ of A respectively below.*

-0.8324367047	-0.2091836735
-0.7454037195	-0.1989795918
-0.6808309565	-0.1887755102
-0.6256758243	-0.1785714286
-0.5761952621	-0.1683673469
-0.5306416864	-0.1581632653
-0.4880148696	-0.1479591837
-0.4476710787	-0.137755102
-0.4091621831	-0.1275510204
-0.3721579654	-0.1173469388
-0.3364042965	-0.1071428571
-0.3016986366	-0.09693877551
-0.2678748608	-0.08673469388
-0.234792892	-0.07653061224
-0.2023324919	-0.06632653061
-0.1703876503	-0.05612244898
-0.1388638181	-0.04591836735
-0.1076743929	-0.03571428571
-0.07673935793	-0.02551020408
-0.0459829574	-0.01530612245

***Table** 1 Correspondence between Input C and Output A.*

*Step 7. We copy **Table** 1 and paste it to a spreadsheet application to get the following scatter plot. The $x-axis$ represents the $x-values$ of the points C on C_1 and the $y-axis$ represents the $x-values$ of the corresponding points A on C_2.*

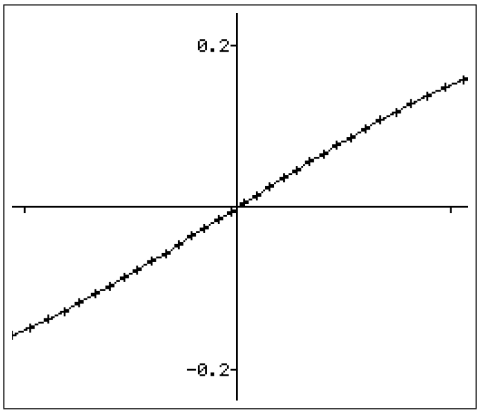

***Figure** 2 Scatter plot between C and A*

We note that the plot above simulates the solution curve for the $x-values$ of C and A respectively, which we will demonstrate here. We define $f(x) = 1 - \sqrt{1-x^2}$ and solve the equation $\left(\frac{g(c)-f(x)}{c-x}\right) \cdot g'(c) + 1 = 0$ explicitly with Scientific WorkPlace 5.5 or Maple 10 to get the expression c in terns of x, which we define it as the function h below:

$$\begin{aligned} h(x) &= \sqrt[3]{\frac{1}{4}x + \sqrt{\frac{1}{27}x^2\sqrt{-x^2+1} - \frac{79}{108}\sqrt{-x^2+1} - \frac{31}{144}x^2 + \frac{185}{216}}} \\ &\quad + \frac{\frac{1}{3}\sqrt{-x^2+1} - \frac{5}{6}}{\sqrt[3]{\frac{1}{4}x + \sqrt{\frac{1}{27}x^2\sqrt{-x^2+1} - \frac{79}{108}\sqrt{-x^2+1} - \frac{31}{144}x^2 + \frac{185}{216}}}} \end{aligned} \tag{3}$$

*We plot $y = h(x)$ below and note that the graph is consistent with **Figure** 2 we got from ClassPad Manager.*

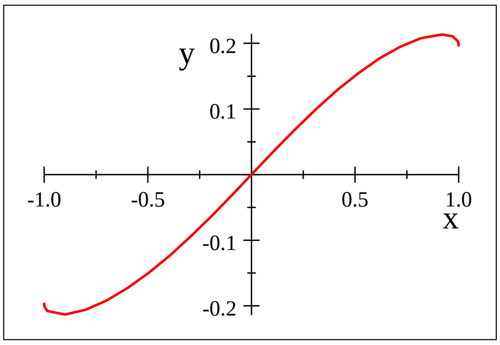

***Figure** 3 Plot between C and A*

*Next we compare following 'partial list' of the numerical computations with **Table** 1 from ClassPad above, they are identical up to four decimal places.*

$$h\left(\begin{pmatrix} -0.8324367047 \\ -0.7454037195 \\ -0.6256758243 \\ -0.6808309565 \\ -0.5761952621 \\ -0.5306416864 \\ -0.4880148696 \\ -0.4476710787 \\ -0.4091621831 \\ -0.3721579654 \\ -0.3364042965 \\ -0.3016986366 \\ -0.2678748608 \\ -0.234792892 \end{pmatrix}\right) = \begin{pmatrix} -0.20919 \\ -0.19899 \\ -0.17858 \\ -0.18878 \\ -0.16838 \\ -0.15817 \\ -0.14797 \\ -0.13776 \\ -0.12756 \\ -0.11736 \\ -0.10715 \\ -9.6947 \times 10^{-2} \\ -8.6743 \times 10^{-2} \\ -7.6539 \times 10^{-2} \end{pmatrix}. \tag{4}$$

2.2 Extremum Distance Between Two Curves in Two Dimensions

Since distance function between two points is a continuous function, if we restrict both functions f and g in a closed and bounded domain, the global maximum and minimum exist by the **Extreme Value Theorem**. By following the idea we described in preceding section, we note that the necessary conditions that the distance AB, where $A = (x, f(x))$ and $B = (c, g(c))$, is the minimum or maximum distance between $y = f(x)$ and $y = g(x)$ are:

$$\begin{aligned} &AB \text{ is perpendicular to the tangent line at } A \text{ and} \\ &AB \text{ is perpendicular to the tangent line at } B. \end{aligned} \tag{5}$$

Or the following conditions must be satisfied simultaneously.

$$\left(\frac{g(c) - f(x)}{c - x}\right) \cdot g'(c) + 1 = 0. \tag{6}$$

$$\left(\frac{g(c) - f(x)}{c - x}\right) \cdot f'(x) + 1 = 0. \tag{7}$$

We demonstrate this by using the following example.

Example 2 *Let* $f(x) = \sin(x) + 5$ *and* $g(x) = -\cos(x - 1)$ *and we* ***restrict our domain*** *in* $[-5, 5]$. *We solve the equations (5) and (6) which result in the following three cases*

1. *When*

$$\begin{aligned} x_1 &= -1.030769192, f(x_1) = 4.142305270, \\ c_1 &= 1.540027135, \ \textit{and}\ g(c_1) = -.8576947300. \end{aligned}$$

We note that the square distance between $(x_1, f(x_1))$ *and* $(c_1, g(c_1)$ *is about* 31.60899376. *We sketch the graphs of* $y = f(x)$, *and* $y = g(x)$, *the vector connecting* $(x_1, f(x_1))$ *and* $(c_1, g(c_1))$, *the tangent line at* $x = x_1$ *and the tangent line at* $x = c_1$ *together as follows:*

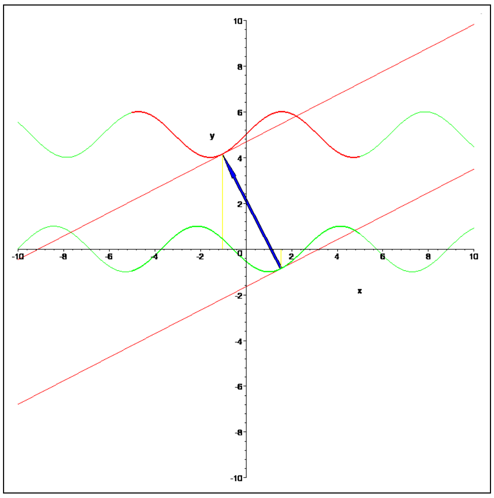

***Figure** 4 Relative Maximum Distance*

2. *When*

$$\begin{aligned} x_2 &= 3.421927911, f(x_2) = 4.723322166, \\ c_2 &= -.8511315841, \ \textit{and}\ g(c_2) = .2766778338. \end{aligned}$$

We note that the square distance between $(x_2, f(x_2))$ *and* $(c_2, g(c_2)$ *is about* 40.49225911. *We sketch the graphs of* $y = f(x)$, *and* $y = g(x)$, *the vector connecting* $(x_2, f(x_2))$ *and*

$(c_2, g(c_2))$, *tangent line at* $x = x_2$ *and tangent line at* $x = c_2$ *together as follows:*

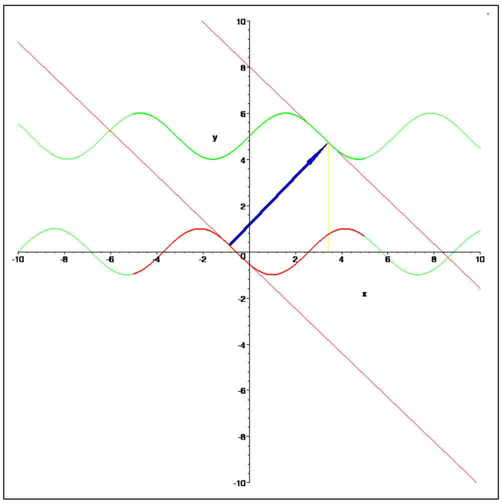

Figure *5 Global Maximum Distance*

3. *When*

$$\begin{aligned} x_3 &= -1.684808257, f(x_3) = 4.006492323, \\ c_3 &= -2.027580723, \text{ and } g(c_3) = .9935076771. \end{aligned}$$

We note that the square distance between $(x_3, f(x_3))$ *and* $(c_3, g(c_3)$ *is about* 9.195569440. *We sketch the graphs of* $y = f(x)$, *and* $y = g(x)$, *the vector connecting* $(x_3, f(x_3))$ *and* $(c_3, g(c_3))$, *tangent line at* $x = x_3$ *and tangent line at* $x = c_3$ *together as follows:*

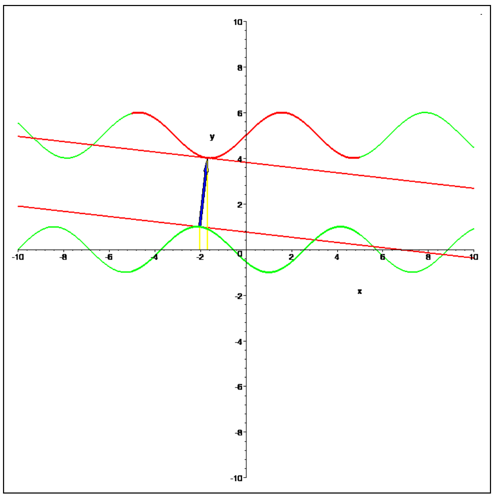

Figure *6 Global Minimum Distance*

We conclude that global maximum distance and minimum distance occur at case 2 and case 3 respectively.

3 Three Dimensional Case

Here we describe how to find the local extremum distance between two surfaces which satisfy

$$f(x, y, z) = 0 \text{ and } g(x, y, z) = 0, \tag{8}$$

where f and g are differentiable functions in its respective domain. Following what we have done in two dimensional case, we see that if $A = (x_1, x_2, x_3)$ is on the surface $f(x, y, z) = 0$ and $B = (y_1, y_2, y_3)$ is on the surface $g(x, y, z) = 0$, the necessary condition for finding such extremum distance is to have

$$\begin{aligned}&AB \text{ is parallel to the normal vector of the tangent plane at } A \text{ and}\\&AB \text{ is parallel to the normal vector of the tangent plane at } B.\end{aligned} \tag{9}$$

More specifically, the following conditions should be met:

$$\begin{aligned}AB &= \lambda_1 (\nabla f) \ \text{ at } A,\\ AB &= \lambda_2 (\nabla g) \ \text{ at } B,\\ f(x_1, x_2, x_3) &= 0, \ \text{ and}\\ g(y_1, y_2, y_3 &= 0.\end{aligned} \tag{10}$$

where $\nabla f = (f_x, f_y, f_z)$ and $\nabla g = (g_x, g_y, g_z)$ are the gradients of f and g respectively. The above set of equations is equivalent to the followings:

$$y_1 - x_1 = \lambda_1 (f_x)_{(x_1,x_2,x_3)}, \tag{11}$$
$$y_2 - x_2 = \lambda_1 (f_y)_{(x_1,x_2,x_3)}, \tag{12}$$
$$y_3 - x_3 = \lambda_1 (f_z)_{(x_1,x_2,x_3)}, \tag{13}$$
$$y_1 - x_1 = \lambda_2 (g_x)_{(y_1,y_2,y_3)}, \tag{14}$$
$$y_2 - x_2 = \lambda_2 (g_y)_{(y_1,y_2,y_3)}, \tag{15}$$
$$y_3 - x_3 = \lambda_2 (g_z)_{(y_1,y_2,y_3)}, \tag{16}$$
$$f(x_1, x_2, x_3) = 0 \text{ and } g(y_1, y_2, y_3) = 0. \tag{17}$$

This gives a geometric interpretation when we apply the Lagrange Multiplier Method in solving this problem. In other words, if our objective is to minimize or maximize the square distance $|\mathbf{x} - \mathbf{y}|^2$ and subject to both $f(\mathbf{x}) = 0$ and $g(\mathbf{y}) = 0$, then we write

$$L(\mathbf{x}, \mathbf{y}, \lambda_1, \lambda_2) = |\mathbf{x} - \mathbf{y}|^2 + \lambda_1 h(\mathbf{x}) + \lambda_1 g(\mathbf{y}) \tag{18}$$

or

$$\begin{aligned}L(x_1, x_2, x_3, y_1, y_2, y_3, \lambda_1, \lambda_2) &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 +\\ &\quad \lambda_1 h(x_1, x_2, x_3) + \lambda_2 g(y_1, y_2, y_3);\end{aligned} \tag{19}$$

Then the necessary condition to achieve the critical distance is to have

$$\nabla L = 0, \tag{20}$$

which will give the same results as our observation from equations (11)-(17). We demonstrate this by using the following example.

Example 3 *Let* $f(x,y,z) = \sin x \cos y - 2 - z$ *and* $g(x,y,z) = x^2 + y^2 - z$*. If we restrict the domain to be* $[-2,2] \times [-2,2]$ *for both functions, and solve* $\nabla L = 0$*. The computation shows only the following solution*

$$\begin{aligned} x_1 &= .9776334541, x_2 := 0, x_3 = -1.170823173, \text{ and} \\ y_1 &= .2794931976, y_2 = 0; y_3 = 0.0.7811644752. \end{aligned}$$

The square distance $|\mathbf{x} - \mathbf{y}|^2$ *is* 2.047249995*, which implies the distance between these two surfaces to be about* 1.430 8*. The graph below shows the solution set will achieve the global minimum distance for these two surfaces.*

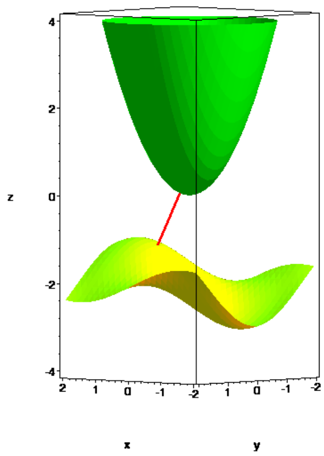

Figure 7 *The shortest distance between two surfaces*

4 Conclusion

Traditionally, when technological tools are not available, students may find applying Lagrange Multiplier Method in solving optimization problems difficult; not only due to the complicated algebraic manipulation nature but also they often do not fully understand the geometric interpretation behind the method. This paper demonstrates that optimization problems can be made interesting if teachers inspire students with geometric motivation. It is also important to note the integration between a dynamic geometry software with a computer algebra system is crucial: the dynamic nature makes analytic geometry **live** and leave **static** and complicated algebraic computations behind the scene for a CAS to do further investigations. Author believes that the integration between a three dimensional dynamic geometry software with a CAS will further make three dimensional problems in Analytic Geometry more accessible.

References

[1] ClassPad Manager, a product of CASIO, *www.classpad.net* or www.classpad.org.

[2] Patoglu, V. & Gillespie, R. B. *Extremal Distance Maintenance for Parametric Curves and Surfaces,* Proceedings of the 2002 IEEE International Conference on Robotics 8 Automation, Washington, DC May 2002, pp2817-2823.

[3] Johnson, D.E. & Cohen, E. *A Framework For Efficient Minimum Distance Computations*, in Proc. International Conference on Robotics and Automation, (Leuven, Belgium), pp. 3678-3684, IEEE, May 1998.

[4] Maple 10, a product of *MapleSoft*, http://www.maplesoft.com.