Designing Applets to Build on Intuition

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Abstract: You have a match box containing an unknown number of matches, and are told that if you had 2 more matches then you would have 9 matches altogether: how many matches are in the box? Even before studying linear equations, most school students correctly answer 7 to this question. However, many students find it difficult to explain clearly how they arrived at the answer – for example, a common response is 'because 2 plus 7 is 9', which is a demonstration that the answer is correct rather than a explanation of the method of solution. In this paper we consider the issues underlying the creation of applets to specifically build on this basic mathematical intuition. One beneficial outcome is that intuition itself is enhanced. We illustrate the general ideas underlying the use of ICT to help develop mathematical skills at the junior level by looking at an applet created specifically to support learners' intuition in solving simple linear equations like the matchbox problem above. The aim is not only to help improve learners' skills in solving more general linear equations, but also to let them become aware that these skills (the appropriate use of the operations of addition, subtraction, multiplication and division) have built upon, and grown out of, their 'natural' intuitive approaches.

1. Introduction

As learners progress through school, and indeed even after school, they build up an intuitive understanding of numbers and arithmetic that enables them to make use of the operations of mathematics to solve everyday problems. To take a very simple example, customers at the supermarket understand that the sum of the price of each of the items purchased will give the total amount to be paid. Although they may not be able to do such addition as effortlessly as the check-out computer, nevertheless the process conforms to some intuitive understanding of number and arithmetic and contains no magic or mysterious elements. One of the key roles of a general mathematics education is to build upon and extend this intuition so that the use of basic mathematical operations and concepts is de-mystified. In this paper we look at the role ICT can play in helping build mathematical intuition.

2. Matchbox Algebra

In this section we will give an overview of the matchbox algebra applet we have developed and discuss its role in building an intuitive approach to the solving of linear equations. (Visit the website `http://www.itee.uq.edu.au/~rduke/MatchboxAlgebra/MatchboxAlgebra.html` if you wish to run this applet.)

Consider the question posed at the top of the screen-shot in Figure 1.

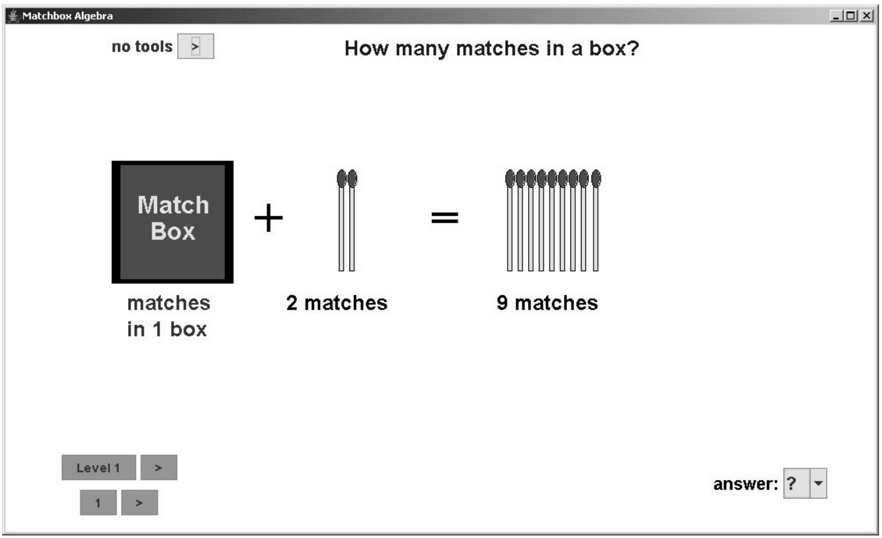

Figure 1

In this first stage of the applet no tools are supplied to help resolve the question – the students must rely entirely upon intuition or prior experience. Students can check if their answer is correct by entering their ‘guess’ (see Figure 2).

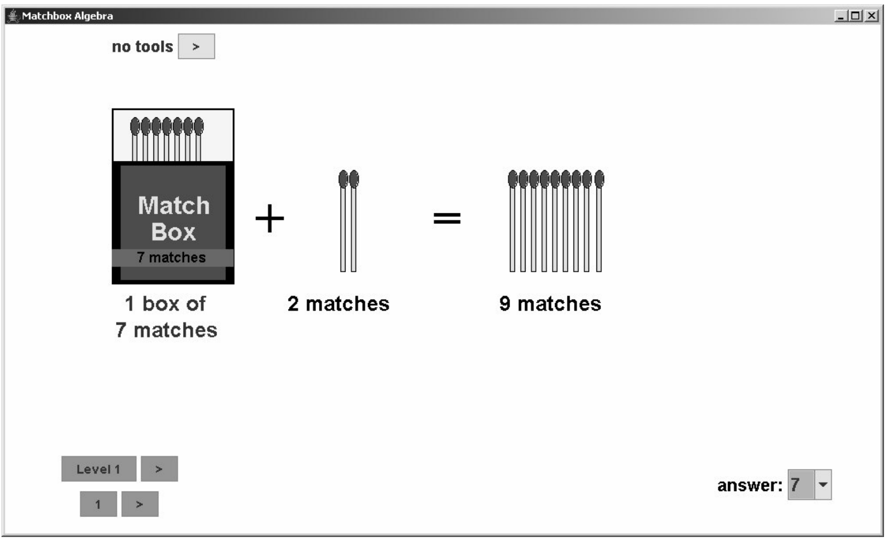

Figure 2

If they are correct, the box opens to reveal its matches; if not, they are urged to try again. When posed such a question, most school students who have reached the stage when they are about to begin the study of linear equations just ‘know’ that the answer is ‘7’. However, often they find it difficult to explain clearly how they arrived at the answer.

As this example illustrates, most students can apply basic mathematical intuition when faced with such a problem. The issue is how to translate that intuition into a systematic method for solving such equations and to ensure that such a ‘way of seeing’ equations is future-proof when they are faced with more difficult questions. Conversely, it is important that they see any such method to be in harmony with their intuition and not magical and imposed. In this example it would obviously be easier to solve the question if the equation were as illustrated in Figure 3.

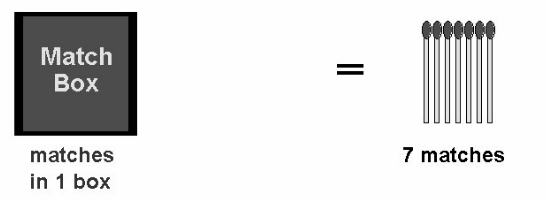

Figure 3

This raises the following issue: can learners manipulate a matchbox equation such as the one in Figure 1 to get it in a form where it is easier to answer the question as to how many matches are in a box? This leads intuitively to the idea of equation simplification and the following two fundamental rules of equation manipulation:

- the equality of total match numbers between the left and right sides of the equation must be maintained, and
- the number of matches in a box is not altered by any such manipulation. In particular, one cannot add or subtract matches to or from a box, and each matchbox for a given question contains the same number of matches.

For example, if the same number of matches or boxes are added or subtracted to both sides of the equation then these rules will hold. In the second stage of the applet the operations of addition and subtraction are provided (see Figure 4) and the students are encouraged to discover what simplifications to the original equation can be made using these operations.

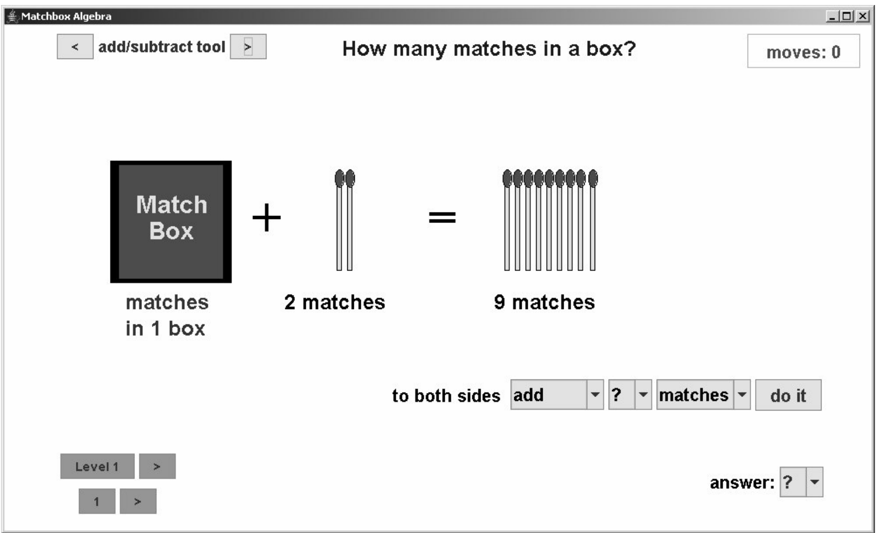

Figure 4

Likewise, if both sides of the equation are divided by the same value then again the above rules for equation manipulation will hold. In the third stage of the applet, the operation of exact integer division is included (see Figure 5). In the case of matchboxes, the number of matches in a box must be a non-negative integer and there is no concept either of a fraction of a match or a negative match. Consequently, all matchbox equations involve only integers and the answer to any question must be a non-negative integer; hence multiplication will not be required when simplifying such equations and only exact integer division need ever be performed.

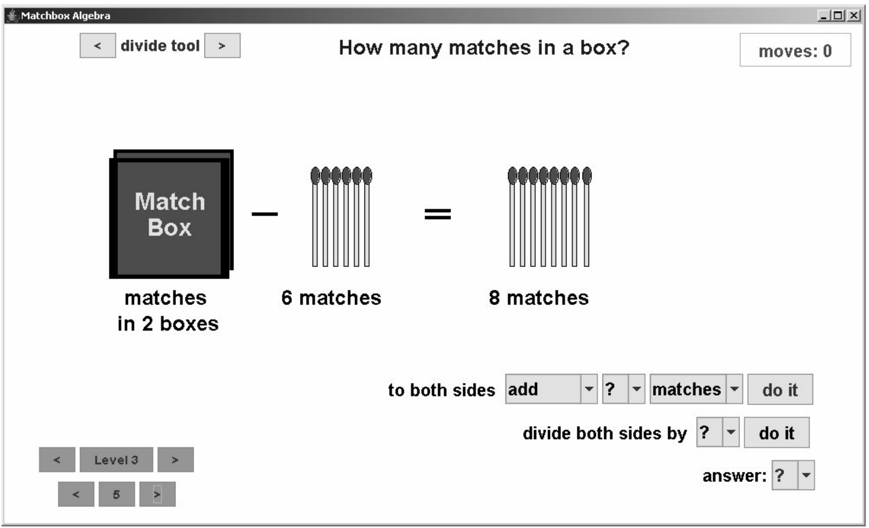

Figure 5

When the student has manipulated a sufficient number of such matchbox equations and is convinced that the operations satisfy the rules for equation manipulation, they are ready to progress to the fourth and final stage of the applet, where the equation is expressed using the more traditional ‘x’ notation (see Figure 6). Here, students are required to manipulate symbols rather than icons and questions could have a fractional or negative answer. Additionally, they are provided with the operation of multiplication as well as division.

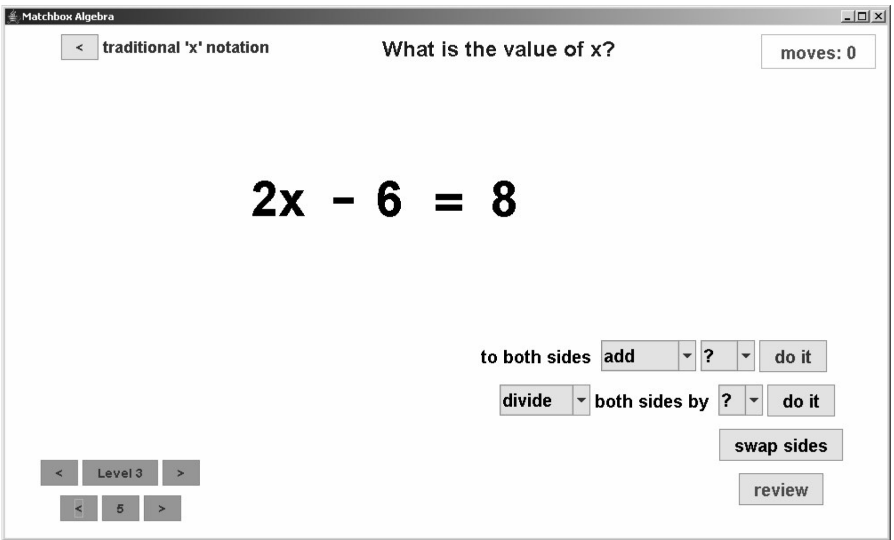

Figure 6

Altogether the applet contains 86 equations partitioned into 6 levels of increasing difficulty. At Level 1 each equation contains only one matchbox and no negative operations; at Level 2 there is still only one matchbox but now there may be negative operations; at Level 3 there can be more than one matchbox, but all the matchboxes are on one side only of the equation; at Level 4 there can be matchboxes on both sides of the equation; at Level 5 larger numbers are used; and finally at Level 6, which can be accessed only from the final fourth stage when the traditional 'x' notation is being used, the equations can have a fractional or negative answer.

3. Connecting the Matchbox and Algebraic Metaphors

The matchbox approach offers students a simple but powerful visual metaphor for understanding and manipulating the components and relationships represented in a linear equation. Within the language of matchboxes and matches, a number of important algebraic principles can be established which can subsequently form the bedrock of students' more formal symbolic understanding. In this section, we provide several examples of some of the algebraic ideas that can first be established in a matchbox form and indicate how these may be later translated to conventional algebraic wisdom.

First, it is clear from that way in which Matchbox algebra problems are set up that, if you are shown a box, you don't know initially how many matches it contains. This can help create the awareness that, in algebra, a letter is often used to represent an as-yet-unknown value. Second, the key question that drives matchbox algebra is, 'Can you work out how many matches are in a box?' The more formal equivalent of such a question might be to realise that equations are often offered to students in order for them to 'solve' them – i.e. to find the value of x that makes the LHS equal to the RHS. A question that concerns students when they start using the Matchbox applet is, 'Where there is more than one matchbox, do they always contain the same number of matches'. This is, in fact, an excellent question but one that may be harder to express formally, where it might take the form, 'When the same letter is used more than once in a given equation, is its numerical value the same?' The other side to this coin is that, when the same letter is used in a *different* equation, it might correspond to a different solution. Students can anticipate this awareness when they quickly discover by tackling different matchbox questions that for each question there is usually a different number of matches in a box.

The metaphor of matchbox algebra can also help prepare students for exploring *methods of solution* of linear equations. For example, it normally requires only a short discussion for students to work out that adding or subtracting matchboxes and matches to or from each side is a useful strategy for simplifying the problem and thereby working out the number of matches in each box. This approach is an excellent precursor to the realisation that adding and subtracting terms from each side of the equation is a useful strategy for simplifying and thereby solving the equation. Finally, and more subtly, using the matchbox approach, students have little difficulty in seeing that simplifying the problem changes it to a different statement but doesn't change the number of matches in each box – it simply makes it easier to work out the answer. It is less obvious, in a more formal context, that simplifying the equation changes the equation but doesn't change the solution – it simply makes it easier to work out the answer.

4. Design Issues

In designing the matchbox algebra applet the authors were particularly influenced by the work of Jerome Bruner [Bruner 1966, 1989, 1996]. Bruner identified three worlds within which learning might take place, which he termed the ‘material’, the ‘imagined’ and the ‘symbolic’. Each world requires different sorts of representations and these, in turn, encourage a corresponding set of three thinking and learning modes, all of which, Bruner believed, were both valuable and mutually complementary. These he referred to as Enactive–Iconic–Symbolic (EIS). He explained his framework in the following terms:

Any domain of knowledge (or any problem within that domain of knowledge) can be presented in three ways: by a set of actions appropriate for achieving a certain result (enactive representation); by a set of summary images or graphics that stand for a concept without defining it fully (iconic representation); and by a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions (symbolic representation). [Bruner, 1966, p. 44]

This framework has been found extremely helpful in working with teachers in the past [Mason and Johnston-Wilder, 2004] and was used to inform the design of the matchbox applet. Specifically, it can act as a reminder to allow learners to work from physical manipulation of apparatus to the use of mental images and then on to symbols. In secondary school mathematics classrooms many teachers move too quickly to symbolisation and the transition is not well understood. Bruner's Enactive-Iconic-Symbolic trio of representations and his suggestions about ways to scaffold transitions may help prompt learners to form and work with conceptual structures.

In working through the matchbox algebra concept, students engage initially with actual closed matchboxes and actual matches (the *Enactive* phase), with the sought-after value being the unknown contents of the matchboxes. However, what can be achieved with actual matches and boxes is limited physically, so students move to the applet (the *Iconic* phase) comprising several levels of sophistication. Within this phase, students work with the highly visual and manipulable screen images. Finally, students shift into the use of the traditional ‘x’ instead of matchboxes (the *Symbolic* phase) and are inducted into expressing mathematical ideas more concisely, leading them towards the use of conventional mathematical symbols.

This is summarised in the following table:

Enactive (E) phase	based on the handling of physical objects (actual matches and boxes) as part of tackling purposeful tasks
Iconic (I) phase	using the matchbox algebra applet
Symbolic (S) phase	moving towards expressing ideas concisely using conventional mathematical symbols.

Another important issue is the need to ensure that students do not get confused between a matchbox and the value (i.e. the number of matches contained in the box) that it represents. When performing the operations of addition and subtraction, a distinction must be made between adding and subtracting matches or boxes. However, at all times it is the number of matches on both sides of the expression that are equated, and for this it is the number of matches in a box that is relevant. To help overcome this common confusion, the wording of the applet images (e.g. see Figures 4 and 5) emphasizes that it is the matches in the boxes rather than the boxes themselves that determine the equation.

5. Using ICT Effectively

There is a basic dilemma for software designers when considering the needs of learners using ICT in the pursuit of mathematical understanding. At one extreme, the software drives the learners (for example, with a programmed learning package offering multi-choice responses) and at the other the learners drive the software (for example, where the software comprises a tool, such as a spreadsheet, that is in the user's control). There are pedagogic issues with each of these extremes. As Hoyles expresses it:

If we want to design investigative environments with computers that will challenge and motivate children mathematically, we need software where children have some freedom to express their own ideas, but constrained in ways so as to focus their attention on the mathematics. [Hoyles, 2004, p. 160]

Hoyles goes on to identify the following three conditions for enabling technology to change pupils' experience of mathematics:

1. *The users of the technology (teachers and students) must appreciate what they wish to accomplish and how the technology might help them;*
2. *The technology itself must be carefully integrated into the curriculum and not simply added on to it;*
3. *The focus of all the activity must be kept unswervingly on mathematical knowledge and not on the hardware or software.* [Hoyles, 2004, p. 161]

These conditions underpin the design and use of the matchbox algebra applets.

Taking Hoyles' first requirement, the authors were aware of the importance that students be given clarity about the nature of the software and the purpose of the tasks. To do this we ensured that the design of the software closely mirrored the actual matchbox activity that they worked with at the 'E' stage. We have tested the matchbox applet with students of various ages in the UK [Graham and Duke, 2006] and in both phases (i.e. using real and virtual matchboxes), the goal of trying to work out how many matches there are in a box was transparent and obvious to the learners.

Hoyles' second concern (stressing the importance of integrating ICT use into the curriculum) is taken on board through placing the teaching sequence within the framework of Bruner's learning modes.

In order to accommodate Hoyles' third point, we designed the software to be used in the classroom with issues of hardware and software kept in the background. In fact, we would go beyond Hoyles' last point and strongly suggest that ICT enables mathematics to be seen as a shared activity and not something done just in isolation. The matchbox applets are designed to run on the interactive whiteboard; students are encouraged to come to the board to share and explain ideas while performing the manipulation operations for all to see. The teacher can often stand back and let the students learn from each other.

6. Conclusions and Future Directions

Feedback from teachers and students strongly suggests that the matchbox approach greatly helps learners grasp and apply the fundamentals of solving linear equations [Graham and Duke, 2006]. All users quickly gained a good understanding of the distinction between boxes and matches and could see that it was the number of matches in a box that determined the equation. These ideas lie at the heart of what is a key stumbling block for many students in their beginning study of algebra – the notion that, when solving an equation, the ‘x’ represents the as-yet-unknown number. Unless this idea is understood, students have only confusion to build on when, later, they are asked questions that require a fuller understanding of what is meant by an algebraic variable.

An extension to the matchbox applet currently under consideration is to provide two types of matchboxes, e.g. red boxes and yellow boxes, where all the red boxes contains the same number of matches, as do all the yellow boxes, but perhaps a different number. This then leads to the idea of simultaneous linear equations.

Other mathematical concepts that the authors plan to tackle with applets include the addition of fractions and the notion of place value. As we have done with matchbox algebra, we will start with the important idea that students come to the study of both of these topics with a set of intuitions that need to be addressed and built upon.

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