# Importance of Linguistic Explanation in Mathematics Education, and the Support from Technology

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**Abstract:** Although black box approach seems to have popularity in high-school mathematics learning in Japan, it certainly devalues the learning of not only mathematics but also engineering subjects that use mathematics. If teachers encourage their students to use a formula as a convenient tool to solve mathematical problem without teaching the hidden mechanism enough, the students tend to memorize it blindly and use it as a black box. They use it in the examination *effectively* and forget it in several months. After that, there remains only vague memory that they have learnt something. In such a learning style, the simple use of technology might cause disaster because it only increases the *effectiveness* of black boxes.

After surveying the low performance of our students in linear algebra, the authors have confirmed the cause as black box approach in their learning of abstract ideas. In the lessons of linear algebra, too many formulas were learnt without understanding the background mechanism and possible applications in the real world. There were few modeling, numerical confirmations, and linguistic explanations of the relations between symbolic & numerical representation and graphic objects. As the result, large number of students applied black box approach especially in linear algebra.

This paper tries to show the importance of linguistic explanation in avoiding black box approach by describing the authors' recent attempt to connect symbolic expressions and graphical representations in linear algebra through linguistic explanation with the help of technology. The further focus on linguistic explanation we have, the lesser black box approach the students apply in their learning, we think.

## 1. Background

Surface learning has been spreading among Japanese high school students [1]. In the learning of algebra, surface learners try to memorize formulas and limited number of solutions as black boxes just before the examinations. They make typical errors most frequently in the calculations of symbolic fractions because of the black box approach. The authors have used their web-based interactive exercise course to find those errors, to let them recognize the importance of rewriting rules, and to guide them to use the rules to think the hidden mechanism of calculations of symbolic fractions [2,3,4]. Although their symbolic calculations have been improved through the activities, the performance in linear algebra haven't. For example, in the achievement tests of high school mathematics for the 3$^{rd}$ grade students (17 years old), performance of the students in linear algebra have been the lowest in our college during the recent years. It was apparently lower than those of symbolic calculations, exponential or logarithmic functions, trigonometry, and calculus, even though the teaching hour is not fewer than them. As the students' success in term-end examinations doesn't guarantee that they have mastered the subject and can solve basic problems one or two years later in the achievement tests, they have to re-learn the fundamental linear algebra when they start to learn college-level linear algebra in their 4th grade.

According to the authors' observation, their low performance in linear algebra was at least accelerated by black box approach in their learning. In the lessons of linear algebra, many formulas were learnt as instant tools for solving problems and the abstractness hinders the students to understand the background mechanism and think of possible applications in the real world. There were few modeling, numeric confirmations, and linguistic explanations of the relations between symbolic & numerical expressions and graphic objects.

In this paper, we would like to focus on the current problem of our students' learning style of linear algebra, discuss a practical approach, and show an example of the help by technology.

## 2. An Example of Black Box Approach in Learning Linear Algebra

In our supplementary lessons of linear algebra, 2nd grade students showed deep dependence on black box approach. Typical example was the calculation of the distance *d* of a plane $ax+by+cz+d=0$ and a point $(x_0, y_0, z_0)$ in the three-dimensional space. All the students selected the formula $d=\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$ and many of them failed to write down the formula properly. Instead, some of them wrote symbolic expressions similar but not equal to the formula. They complained that the formula was difficult to remember because it was too complicated and had similarity to several other formulas. No one could explain the meaning of the formula with their own words or could deduce it from the vector equation of the plane and the position vector of the point. Adding to that, most students could not explain the graphical image of dot product of two vectors, nor deduce the vector equation of a plane. It was difficult for them to separate a vector equation of a plane from one of a straight line in the three dimensional space. All of them said that linear algebra is difficult to learn because they have to remember a vast amount of formulas blindly to solve the problems.

Apparently, their learning style is the cause of the difficulty. In the above problem, one of the simplest explanation of the distance *d* is the dot product of a unit perpendicular vector to the plane $\overrightarrow{n_0}=\dfrac{(a \quad b \quad c)}{\sqrt{a^2+b^2+c^2}}$ and the vector drawn from the point $(x_0, y_0, z_0)$ to an arbitrary selected point $(x, y, z)$ on the plane: $\vec{x}-\overrightarrow{x_0}=(x-x_0, y-y_0, z-z_0)$. The graphical image of the dot product, especially of the unit vector and another vector, is the key to understand the above equation.

When one of the authors explained how to express a straight line in two types of vector equations in the two dimensional space; $\vec{x}=t\cdot\vec{a}+\vec{b}$ and $\vec{x}\cdot\overrightarrow{n_0}=d$, it took two hours for the students to understand the graphical image of the equations. After the explanation of the latter equation, where a unit vector $\overrightarrow{n_0}$ is the perpendicular vector to the line and the distance *d* between the line and the origin play the important role, some of them finally recognized the importance to think the graphical image of a vector equation and dot product. After the explanation, the discussion in the three dimensional space was rather easy for the students to accept with a help of 3D graphics produced by CAS.

## 3. An Example Introduction of Vector Equation and Dot product

After the supplementary lesson of linear algebra after the regular lessons, we confirmed the importance and need of linguistic explanation, especially the relation between graphic objects and symbolic expressions in linear algebra. And we built an introduction lesson of vectors starting with a realistic situation. Because too abstract explanation was a reason of students' difficulties in learning vectors, we tried to explain from an application through the need of a mathematical model to the definition of vector operations. Our application and the explanation is as follows:

You are an engineer to design a robot to guide travelers walking through an old city. The robot will have the coordinates of the starting point, the goal, the current position, and next accessible points on the map (2D plane). All the paths connecting the accessible points are straight lines. Your customer (a travel agency) requests you to design the robot to select minimum number of paths and shortest traveling distance. Because the robot only knows the coordinates of next accessible points but not all the points on the map, it needs a measure for selecting the next optimum path among the candidates.

Figure 1 shows an example application. Your guest (a traveler) wants to arrive at a point where he can cross the river with a boat (the goal). The map of the city shows your robot two possible routes, each composed of four paths. Which path should it select at the starting point? How to evaluate each path, what characteristics of the path do you propose to measure for your robot?

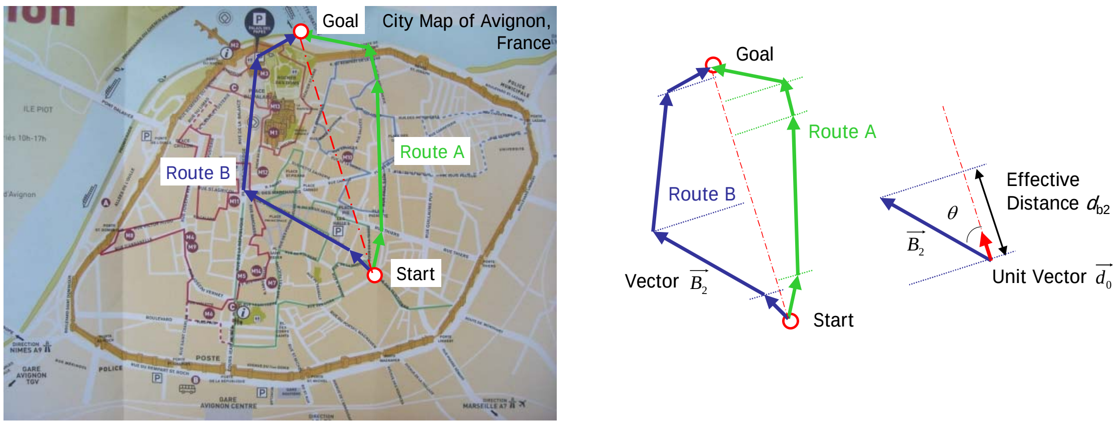


Fig. 1 Vectors as walking paths and their contribution to the traveling

One possible measure is the distance. Although the distance from the starting point to the goal is independent of the selected route, each path on the route has its own distance, direction, and its contribution to the traveling, which we call the effective distance of the path. Let's try to evaluate these distances with the help of vectors.

When we want to describe the distance from the starting point to the goal from the coordinates of the starting point $(x_s, y_s)$ and the goal $(x_g, y_g)$, we can define a vector $\vec{d} = (x_g - x_s \quad y_g - y_s)$, where the absolute value $d = \left|\vec{d}\right| = \sqrt{(x_g - x_s)^2 + (y_g - y_s)^2}$ is the straight distance (length) of the two points and the direction from the initial point (starting point) to the terminal point (goal) in the two-dimensional plane (the angle rotating counter clock wise from the east) is $\alpha = \tan^{-1}\dfrac{y_g - y_s}{x_g - x_s}$. If we describe the length in kilometers, we need a unit vector, which length is 1 kilometer and the direction is the same as $\vec{d}$. The unit vector should be defined as $\overrightarrow{d_0} = \dfrac{\vec{d}}{|d\,|}$.

Next, we define each path of the route B in vectors $\overrightarrow{B_1}, \overrightarrow{B_2}, \overrightarrow{B_3}, \overrightarrow{B_4}$ according to the order to be taken, where each vector expresses its length and direction. And route B is expressed as the addition of those vectors: $\vec{B} = \overrightarrow{B_1} + \overrightarrow{B_2} + \overrightarrow{B_3} + \overrightarrow{B_4}$ from the coordinates of given points on the map. We can explain the practical role of vectors in expressing each path of a route. Route A and B become equal vectors after the simplification but composed of different combination of path-vectors. All possible routes could be expressed as additions of path-vectors and be evaluated mathematically.

With this introduction lesson, most of the students understand the addition of vectors fairly easily, but for the effective distance we need more explanation on the characteristics of unit vectors, the definition of dot product, and the effective distance as the combined result of them. Those are as follows:

Because the robot only receives the coordinates data of the next accessible points, for example $A_1$ and $B_1$ at the starting, it needs a measure for selecting the optimum path only from the limited information of path vectors.

The length of each path, for example, the length of second path of route B is defined as the absolute value of the vector $\left|\overrightarrow{B_2}\right|$, and its direction should be evaluated its difference from vector $\overrightarrow{d_0}$. Then the effective distance $d_{b2}$, the contribution of $B_2$ to the entire traveling, is defined as $d_{b2} = \left|\overrightarrow{B_2}\right| \cos\theta = \overrightarrow{B_2} \cdot \overrightarrow{d_0}$. A unit vector is the vector $\overrightarrow{d_0}$ whose absolute value is always 1 ($\left|\overrightarrow{d_0}\right| = 1$). Because the absolute value is always the same, the only change of a unit vector is its direction. If we define the direction in the two-dimensional plane as $\alpha$ : the angle rotating counter clock wise from the east, the unit vector is expressed as $\overrightarrow{d_0} = (\cos\alpha \quad \sin\alpha)$.

If we express the path $B_2$ in a vector, whose absolute value is $B_2 = \left|\overrightarrow{B_2}\right|$ and the direction as the angle $\theta$ from the unit vector in counter clockwise, it becomes

$\overrightarrow{B_2} = B_2(\cos(\alpha+\theta) \quad \sin(\alpha+\theta))$.

Then the dot product of the vectors is $\overrightarrow{B_2} \cdot \overrightarrow{d_0} = B_2(\cos(\alpha+\theta) \quad \sin(\alpha+\theta)) \cdot (\cos\alpha \quad \sin\alpha)$

$= B_2\{\cos(\alpha+\theta)\cdot\cos\alpha + \sin(\alpha+\theta)\cdot\sin\alpha\} = B_2\cos(\alpha+\theta-\alpha) = B_2\cos\theta$ . It expresses the effective distance $d_{b2}$ of $B_2$ along the direction from the initial point to the terminal point.

Here we could explain the practical role of dot product as effective distance in this application and the reason of defining a unit vector to construct a mathematical measure for the robot. One possible instruction for the robot is to select the next path of maximum effective distance.

## 4. Technology Helps

Technology helps to display the geometric applications of vectors and vector operations. It is especially useful to show them in three dimensions just in front of the students with reality. 3D graphic object, which could be rotated on the students' screen display, shows the equality of vector operations in 2D and 3D. 3D graphics is important because by dealing with geometric objects in 3D, students finally recognize the usefulness of vectors, where dimension independent operations are

valid in 2D and 3D and other operations without vectors are too complicated to conduct in 3D. When the students recognize the usefulness of vector operations in 3D, they are motivated to learn vectors also in 2D settings as the simplified examples of 3D objects.
For example, vector operations used in the application of walking paths through an old town is also valid in three-dimensional applications (Fig. 2). Every route from the starting point to the goal in 3D space is also expressed as the sum of path-vectors, and the effective distance of each path is evaluated by dot product of the path-vector and unit direction vector $\overrightarrow{d_0}$. The only difference in 3D is that all vectors have three components instead of two.
Adding to that, the effective distance calculated by the dot product is exactly the distance of a plane and a point in 3D space, which is exactly the same theme as described in section 2. So we could explain the mechanism of the formula of calculating distance by starting from an application, which is more acceptable to engineering students.

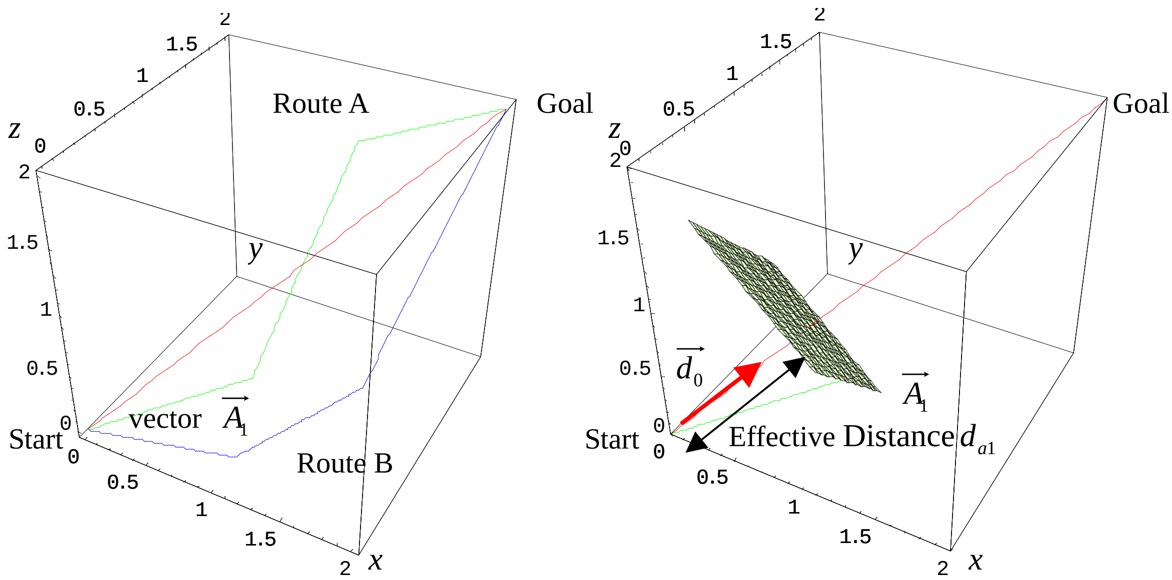


Fig. 2 Walking paths in 3D paths and their contribution to the traveling

## 5. Discussion

We compare the approach taken in this paper with the current major approach in our college (Table 1). Major weakness of the current approach is apparently lower motivation of the students to learn vectors.
Students are not motivated to learn vectors because:
1) Few applications in the real world are connected to vectors in their lessons. Students feel the wide gap between mathematical model of vectors and their life.
2) Because they are used to functional representation of graphic object in 2D, they tend to see vectors and the operations as trivial techniques.

To overcome the first possible reason, we should start our instruction of vectors with an application. After the confirmation that a mathematical model is necessary to solve the actual problem effectively, we could introduce vectors and the operations.

Table 1 Comparison of current and proposed approach of vectors and the operations

| | Current Major Approach | Proposed Approach |
|---|---|---|
| Direction of instruction | Definition, theorems, and lastly their applications | From applications via the need of mathematical model to the definition |
| Role of formulas | Instant tools of problem solving, which students need to memorize | Summarized expression of engineering solution used in the applications |
| Explanation | Symbolic expressions as definition and their rewriting rules | Linguistic explanation explaining the relation of graphic objects and symbolic expressions |
| Technology | Hardly used | Graphics (including 3D) are used at introduction, modeling, and confirmation |

Although we usually let them think and calculate in 2D, we should also show 3D applications in early stage of our lessons. Then the students recognize the merit of using vectors as the dimension independent procedure of vector operations even their current operations are done in 2D.

## 6. Conclusion

In teaching linear algebra of high school level, the authors used the teaching approach, starting from an actual application, emphasizing the need of a mathematical modeling, and finally to mathematical definition of vectors and the operations. In this approach, linguistic explanation plays the major role in class and 3D graphics helps the students to feel the reality of the mathematical model. We expect the approach to improve the performance of our students in linear algebra, and we gain the evidence in the near future.


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