The Third Motivation for Spherical Geometry

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Abstract: Historically, spherical geometry has developed mainly on the terrestrial globe and the celestial globe. In this paper, we will introduce the third globe "*visual globe*" centered at our eyes. When we see the external world, we are under an illusion that our view screen is a plane. Locally it is true, however, globally it is more natural to think that our view screen is a sphere centered at our eyes. In this concept, we will study "*visual angles*" of a rectangle in the space. An angle in the space changes its *visual angle* according to our viewpoint. This *visual angle* is easily measured on the *visual globe* and there is a very simple relation among the *visual angles* of a rectangle. We will also realize the *visual angle* on a two dimensional Euclidean plane along with the dynamic geometry software *Cabri II Plus*. Furthermore, we will find out that the simple relation above is also true in hyperbolic geometry.

1. Introduction

An angle in the three dimensional Euclidean space changes its appearance according to our viewpoint. This appearance is also one of information about a certain relation between the angle and the viewpoint [5-7]. To analyze this information, we will introduce two ideas: *visual angle* and *visual globe*. Intuitively, the visual globe is our view screen centered at the viewpoint, and on this globe we can measure visual angles as the angle of a spherical triangle. In this paper, we will focus on the four visual angles of a rectangle in the space. Using spherical trigonometry, we will find out a simple relation among them.

The definition of visual angle and a representation of the visual angle on the visual globe are described in Section 2. We study the relation among the visual angles of a rectangle in Section 3. In Section 4, we will introduce how to realize visual angles in a plane along with a dynamic geometry software. The stereographic projection plays an important role. This realization proposes an educational approach to spherical geometry without difficult calculation such as spherical trigonometry. Furthermore, we will try to extend the idea of visual angle to hyperbolic geometry in Section 5. The relation among the visual angles of a rectangle is also true. Using the Poincare model, we can easily confirm it by a simple construction.

2. Visual Angle on the Visual Globe

In this section, we will introduce two important ideas: visual angle and visual globe. Let us start from the definition of visual angle.

Definition 2.1(Visual Angle) Let $\angle BAC$ be a fixed angle determined by three points A, B and C in the three dimensional Euclidean space $\mathbf{E}^3$ (Figure 2.1). The *visual angle* from a viewpoint O is defined as the dihedral angle of the two faces OAB and OAC of the tetrahedron $OABC$.

The following proposition shows that the visual globe is useful to measure visual angles.

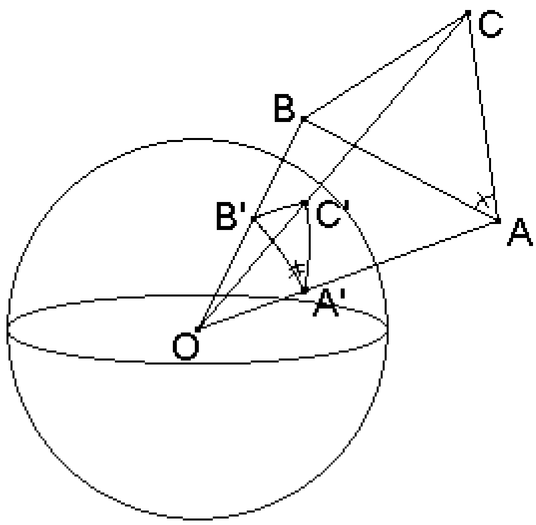

Figure 2.1 Visual angle on the visual globe.

Proposition 2.1(Visual Globe) Let *A'*, *B'* and *C'* be the projected points of *A*, *B* and *C* from the viewpoint *O* onto the (unit) sphere centered at *O* (Figure 2.1). Then the visual angle of $\angle BAC$ is equal to the angle $\angle A'$ of the spherical triangle $\Delta A'B'C'$.

Proof. Let $\vec{V}$ be a vector tangent to the arc *A'B'* at *A'*, and $\vec{W}$ a vector tangent to *A'C'* at *A'*. The angle between $\vec{V}$ and $\vec{W}$ is equal to $\angle A'$. On the other hand, since both vectors $\vec{V}$ and $\vec{W}$ are perpendicular to the line *OA*, the angle between $\vec{V}$ and $\vec{W}$ is also equal to the dihedral angle of the two faces *OAB* and *OAC*. ■

In this way, it turns out that it is better to regard our view screen as a sphere centered at our viewpoint. Of course, this screen is locally Euclidean and when we look outside in the small area, it is enough to regard our view screen as a plane. This visual globe is the third motivation for spherical geometry. When we look up the ceiling in the room, each corner has the visual angle greater than $90°$. Why the sum of four angles of a rectangle is greater than $360°$? The reason is simple, because our view screen is not a plane but a sphere. In the following argument, we use only one spherical trigonometry, that is, the spherical cosine law for angles ([4] p.59):

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

where *a* is the opposite side of the vertex *A* in the spherical triangle ΔABC.

3. Visual Angles of Rectangle

In this section, let us consider the relation among four visual angles of a rectangle in the space. Figure 3.1 shows the central projection of a rectangle in the space onto the visual globe. Without loss of generality, let us assume that a rectangle is arranged on the plane *z*=constant and two pairs of opposite sides are parallel to x-axis and y-axis, respectively. Then four projected sides are great circles on the sphere passing through *X=(1,0,0), –X=(–1,0,0), Y=(0,1,0)* or *–Y=(0, –1,0)*. These points correspond to the vanishing points of four sides.

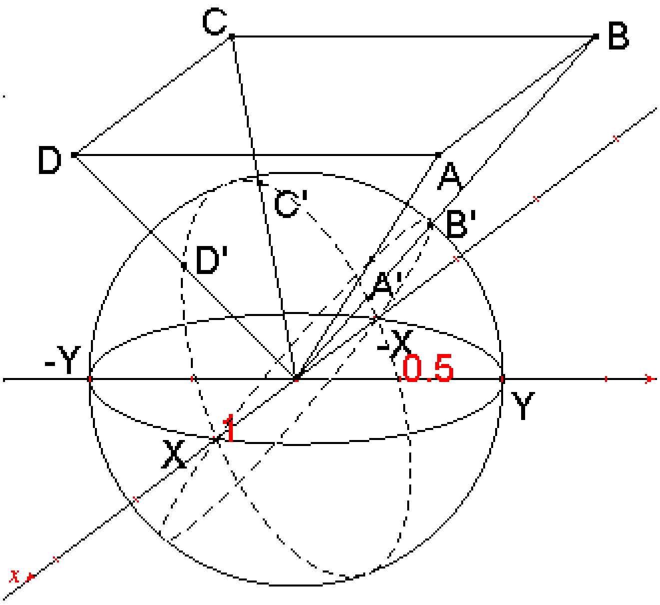

Figure 3.1 Central projection of a rectangle on the visual globe.

Theorem 3.1(Visual Angles of Rectangle) Four visual angles *A'*, *B'*, *C'* and *D'* of a rectangle *ABCD* in the space satisfy the following simple relation:

$$\cos A'\cos C' = \cos B'\cos D'.$$

Proof. Proposition 2.1 enables us to consider the visual angles on the visual globe as in Figure 3.1. Note that two great circles *A'B'* and *C'D'* pass through *X* and –*X*. In the same way, two great circles *B'C'* and *D'A'* pass through *Y* and –*Y* (see, Figure 3.2). Let

$$\alpha = \angle A'XY = \angle B'(-X)Y, \quad \beta = \angle B'Y(-X) = \angle C'(-Y)(-X),$$

$$\gamma = \angle C'(-X)(-Y) = \angle D'X(-Y), \quad \delta = \angle D'(-Y)X = \angle A'YX.$$

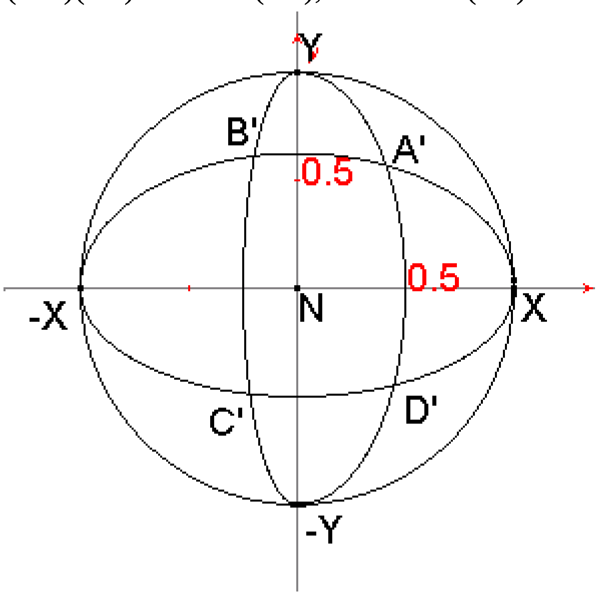

Figure 3.2 Orthogonal projection of a spherical quadrangle from the North Pole.

Appling the spherical cosine law for angles to the spherical triangle *A'XY*,

$$\cos A' = -\cos\alpha\cos\delta + \sin\alpha\sin\delta\cos XY .$$

Since $XY = \pi/2$,

$$\cos A' = -\cos\alpha\cos\delta .$$

In the same way, the following four equations are derived in total:

$$\cos A' = -\cos\alpha\cos\delta, \quad \cos B' = -\cos\beta\cos\alpha,$$

$$\cos C' = -\cos\gamma\cos\beta, \quad \cos D' = -\cos\delta\cos\gamma.$$

Now the equation $\cos A'\cos C' = \cos B'\cos D'$ is trivial. This completes the proof. ■

Remark 3.1 The sum of four visual angles of a rectangle represents the ratio of occupied field of vision to the visual globe, that is ([3] pp.278-279, [4] p.51):

$$\text{(the field of vision of a rectangle } ABCD\text{)} = A' + B' + C' + D' - 2\pi .$$

For example, in the case of a huge rectangle, all visual angles are nearly equal to π, so the field of vision is almost 2π, that is, the area of the hemisphere. On the other hand, in the case that a rectangle is very small for the observer, the sum of four visual angles is nearly equal to 2π, so the field of vision is nearly equal to 0.

4. Realization of Visual Angle in the Plane

In this section, we will introduce a simple method how to realize visual angle in the plane. We cannot measure the exact visual angle by the orthogonal projected image as in Figure 3.2. To realize the exact visual angle, we use the stereographic projection from the South Pole to the xy-plane. This projection has very important properties: conformality and circle-to-circle correspondence. As in Figure 4.1, four projected great circles are circles (arcs) passing through vanishing points *X*, *–X*, *Y* and *–Y* on the equator. These four vanishing points are fixed under the stereographic projection. Drawing arbitrary four arcs passing through four points *X*, *–X*, *Y* and *–Y*, we can easily check the relation $\cos A'\cos C' = \cos B'\cos D'$ by measuring the four angles *A'*, *B'*, *C'* and *D'* in the plane.

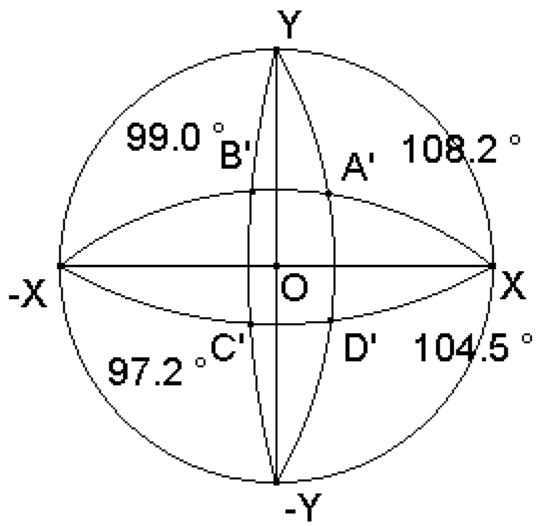

Figure 4.1 Stereographic projection of a spherical quadrangle.

5. Geodesic Rectangle in Hyperbolic Geometry

In this section, we will try to apply the result above to another geometry, hyperbolic geometry. To do this, let us introduce an idea of geodesic rectangle.

Definition 5.1(Geodesic Rectangle) A quadrangle *ABCD* is called a *Geodesic Rectangle* if and only if there are two orthogonal geodesics g_1 and g_2 such that $AB \perp g_1, CD \perp g_1, BC \perp g_2$ and $DA \perp g_2$.

Remark 5.1 In Euclidean geometry, a rectangle is a geodesic rectangle, however, two orthogonal geodesics are not unique.

The following theorem characterizes geodesic rectangle in spherical geometry.

Theorem 5.1(Geodesic Rectangle in $\mathbf{S}^2$) Let *A*, *B*, *C* and *D* be four non-collinear points in $\mathbf{S}^2$. A quadrangle *ABCD* in $\mathbf{S}^2$ is a geodesic rectangle if and only if $\cos A \cos C = \cos B \cos D$.

Proof. If a quadrangle *ABCD* in $\mathbf{S}^2$ is a geodesic rectangle, we can set up a system of coordinates with the intersection of orthogonal geodesics at the North Pole as in Figure 3.2. Using the same technique in the proof of Theorem 3.1, it is easy to show that a geodesic rectangle satisfies $\cos A \cos C = \cos B \cos D$.
To prove the converse, let us show that there exists two orthogonal geodesics as in Definition 5.1. Let *P* be the nearer point to *D* of $AB \cap CD$ and *Q* the nearer point to *D* of $BC \cap DA$. First, let us show $PQ = \pi/2$. In fact, by the law of cosines for angles,

$$\cos A = -\cos\alpha\cos\gamma + \sin\alpha\sin\gamma\cos\ell, \quad \cos B = -\cos\alpha\cos\delta + \sin\alpha\sin\delta\cos\ell,$$

$$\cos C = -\cos\beta\cos\delta + \sin\beta\sin\delta\cos\ell, \quad \cos D = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\ell,$$

where $\alpha = \angle APQ$, $\beta = \angle DPQ$, $\gamma = \angle DQP$, $\delta = \angle CQP$ and $\ell = PQ$. The equation $\cos A \cos C = \cos B \cos D$ and a direct computation imply that $\cos\ell \sin(\alpha-\beta)\sin(\delta-\gamma) = 0$, that is $PQ = \pi/2$. Take a point *R* as one of poles of the geodesic *PQ*, then the spherical triangle *PQR* is a right regular spherical triangle. Regarding *Q* as the North pole and *PR* as the equator, the longitude *DA* passing through *Q* is perpendicular to the equator *PR*. In this way, it is found that *PR* and *QR* are two orthogonal geodesics of the quadrangle *ABCD*. This completes the proof. ■

Now, let us consider geodesic rectangle in hyperbolic geometry $\mathbf{H}^2$. Here just recall the following proposition by Lambert without proof (see, [1] pp. 156-157).

Proposition 5.1(Lambert quadrilateral) Let *OABC* be a quadrangle in $\mathbf{H}^2$. If $O = A = C = \pi/2$ (Figure 5.1), then

$$\cos B = \sinh OA \sinh OC.$$

Remark 5.2 In spherical geometry, there is a similar equation. Let *OABC* be a quadrangle in the unit sphere $\mathbf{S}^2$. If $O = A = C = \pi/2$, then

$$\cos B = -\sin OA \sin OC.$$

This fact is directly derived from the proof of Theorem 3.1.

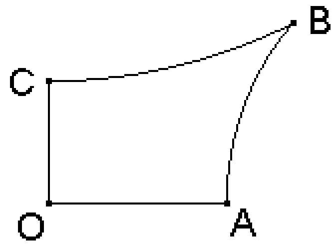

Figure 5.1 Lambert quadrilateral.

Theorem 5.2(Geodesic Rectangle in $\mathbf{H}^2$) If a quadrangle *ABCD* in $\mathbf{H}^2$ is a geodesic rectangle, then four angles satisfy the following simple relation:

$$\cos A \cos C = \cos B \cos D.$$

Proof. We can see that a geodesic rectangle is composed of four Lambert quadrilaterals as in Figure 5.2. Let $P = AB \cap g_2, Q = BC \cap g_1, R = CD \cap g_2, S = DA \cap g_1$ and $O = g_1 \cap g_2$. Applying Proposition 5. 1 to each Lambert quadrilateral, one has

$$\cos A = \sinh OS \sinh OP, \quad \cos B = \sinh OP \sinh OQ,$$

$$\cos C = \sinh OQ \sinh OR, \quad \cos D = \sinh OR \sinh OS.$$

Now the equation $\cos A \cos C = \cos B \cos D$ is trivial. This completes the proof. ■

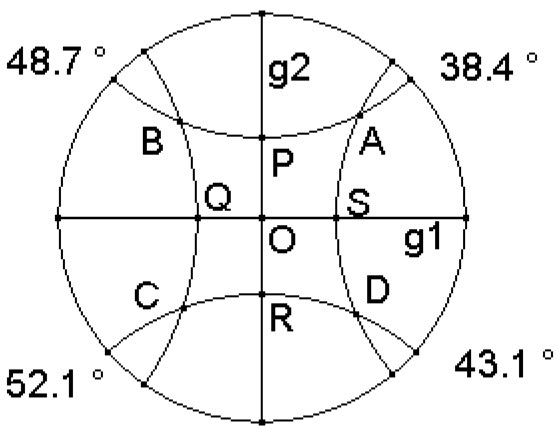

Figure 5.2 Geodesic rectangle in the Poincare model.

In this way, the simple idea of visual angle leads us to not only spherical geometry but also hyperbolic geometry. In addition, it turns out that spherical and hyperbolic geometries are connected by the idea of geodesic rectangles. At the end of this paper, let us introduce a few interesting properties between spherical and hyperbolic geometries from the aspect of complementary angle.

Example 5.1(Complementary Angles in the Circle) Let *A* be an arbitrary point in the unit circle as in Figure 5.3. First, regarding this point *A* as a stereographic projected point on the unit sphere $\mathbf{S}^2$, draw two arcs *XA(–X)* and *YA(–Y)* where *X=(1,0), –X=(–1,0), Y=(0,1)* and *–Y=(0, –1)*. Let α be the angle between the arcs *XA* and *YA*. In the next, regarding the point *A* as a point in the Poincare model of $\mathbf{H}^2$, draw two geodesics *AP* and *AQ* perpendicular to *X(–X)* and *Y(–Y)* , respectively. Let β be the angle between the arcs *AP* and *AQ*. Then two angles α and β always satisfy that $\alpha + \beta = \pi$. One can easily check this property by construction.

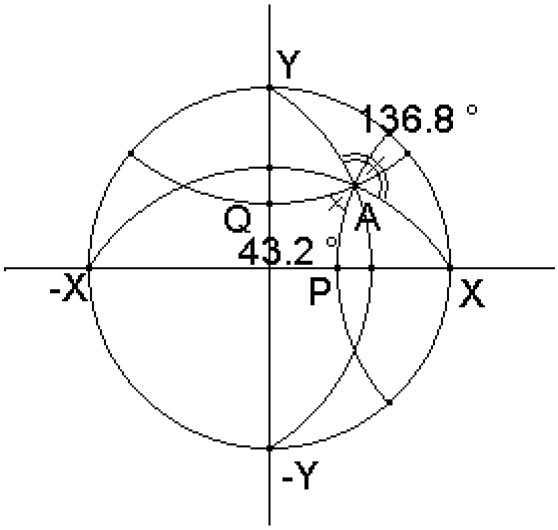

Figure 5.3 Complementary angles in spherical and hyperbolic geometries.

Example 5.2(Complementary Geodesic Rectangles) For an arbitrary geodesic rectangle *ABCD* in $\mathbf{S}^2$, there exists a geodesic rectangle *A*B*C*D** in $\mathbf{H}^2$ such that *A+A*=B+B*=C+C*=D+D*=* π as in Figure 5.4. The areas of these two rectangles are equal. In fact,

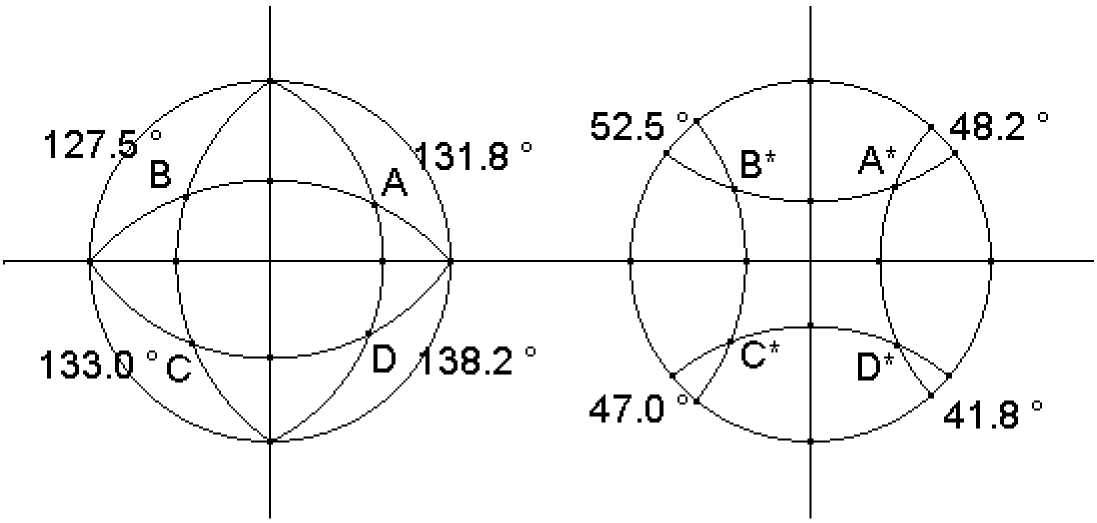

Figure 5.4 Complementary geodesic rectangles.

$$\begin{aligned} area(A*B*C*D*) &= 2\pi - (A*+B*+C*+D*) \\ &= (\pi - A*) + (\pi - B*) + (\pi - C*) + (\pi - D*) - 2\pi \\ &= A + B + C + D - 2\pi \\ &= area(ABCD). \end{aligned}$$

The construction of complementary geodesic rectangle is a little bit complicated. Figure 5.5(left) shows a part of Figure 5.4. The arc *AD* is a geodesic of $\mathbf{S}^2$ perpendicular to x-axis (one of orthogonal geodesics) at *S*. On the other hand, the arc *A*D** is a corresponding geodesic of $\mathbf{H}^2$ perpendicular to x-axis at *H*. The idea is to make the point *H* in $\mathbf{H}^2$ from *S* in $\mathbf{S}^2$ such that $\sin OS = \sinh OH$. Note that *OS* and *OH* are measured by spherical and hyperbolic metrics, respectively. If we identify $\mathbf{R}^2$ with **C**, these metrics are given as ([2] p. 29 and p. 60)

$$ds = \frac{2|dz|}{1+|z|^2} \text{ and } ds = \frac{2|dz|}{1-|z|^2}.$$

Let *S=(a,0)* and *H=(b,0)* in Euclidean coordinates. Then

$$OS = \int_0^a \frac{2dz}{1+|z|^2} = 2\tan^{-1} a, \quad OH = \int_0^b \frac{2dz}{1-|z|^2} = \log\left(\frac{1+b}{1-b}\right),$$

therefore, $\sin OS = \dfrac{2a}{1+a^2}$ and $\sinh OH = \dfrac{2b}{1-b^2}$. The equation $\sin OS = \sinh OH$ implies that

$$\frac{a}{1+a^2} = \frac{b}{1-b^2}.$$

Figure 5.5(right) shows the construction of *H* from *S* :

$$S = (a,0) \;\rightarrow\; P_1 = \left(\frac{2a}{1+a^2}, \frac{1-a^2}{1+a^2}\right) \;\rightarrow\; P_2 = \left(\frac{2a}{1+a^2}, 1\right) = \left(\frac{2b}{1-b^2}, 1\right) \;\rightarrow$$

$$P_3 = \left(\frac{2b}{1+b^2}, \frac{1-b^2}{1+b^2}\right) \;\rightarrow\; H = (b,0).$$

Using this technique, we can easily construct a pair of complementary geodesics.

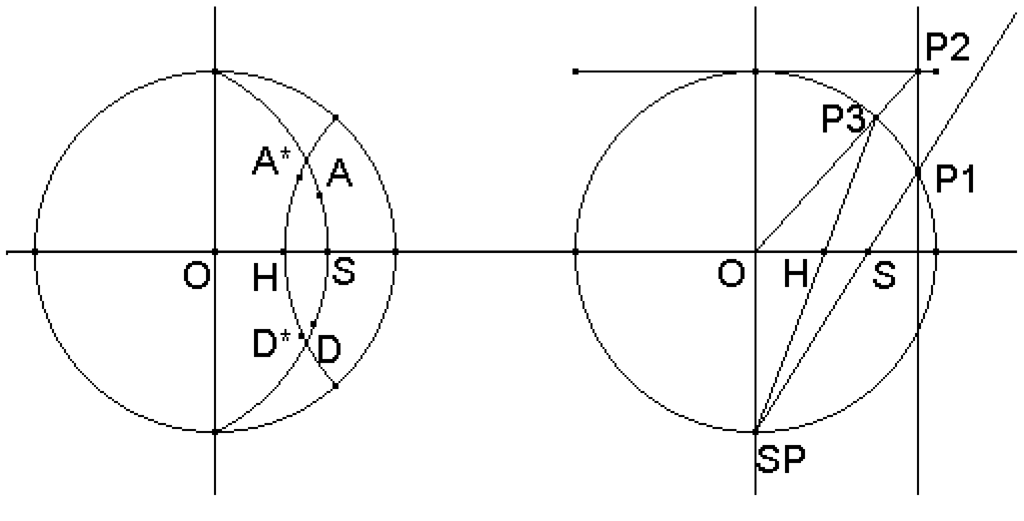

Figure 5.5 Construction of H in $\mathbf{H}^2$ from S in $\mathbf{S}^2$.

6. Conclusion

In this paper, we have proposed one of natural approaches to not only spherical geometry but also hyperbolic geometry. *Visual angle* is the start point. *Visual globe* is our view screen and the *visual angle* is measured on this sphere. Four visual angles of a rectangle in the space have a simple relation, which is extended to hyperbolic geometry. On the other hand, stereographic projection enables us to construct the visual angle on a plane.

Here is a simple question: what is the mean of visual angle in hyperbolic geometry? This natural question is yet to be investigated. We have already found out a few interesting things, complementary angle and complementary geodesic rectangle. These things may give us a clue for a deep understanding and a further study of spherical and hyperbolic geometries.

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