The Center of Gravity of Classes of Cylindrical Solids via a Computer Algebra System

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Abstract: The paper discusses the center of gravity of several classes of right-cylindrical solids. As the base of the solid, one can consider different shapes, such as circles, rectangles, or astroids. Each solid considered has a fixed side, but a variable top, which acts as the roof of the solid. As the roof changes in some prescribed manner, the center of gravity G of the solid changes. Thus one can consider the problem of finding the locus of this center of gravity in space. A computer algebra system (CAS) such as *Mathematica* is a useful tool to study these locus problems. *Mathematica* not only helps with calculating tedious triple integrals involved in solving the problem, but also helps visualize the locus of G via animation techniques. The paper also includes some theorems describing the behavior of the center of gravity G as the solid changes.

1. The Center of Gravity of a Right-Elliptic Cylinder bounded by a Plane

Consider the right elliptic cylinder E_1 in three dimensions given by the equation $x^2/a^2 + y^2/b^2 = 1$, where a and b are fixed positive constants. Let $P(0,0,c)$ be a fixed point on the z-axis where $c > 0$ is a constant. Let $\mathcal{P}$ be the plane through the point P with variable normal vector $< s,t,1 >$ where s and t are real parameters. It is then clear that the equation of the plane $\mathcal{P}$ is given by (see [11] and [13])

$$z = c - xs - yt \tag{1.1}$$

We will assume that s and t are such that the plane $\mathcal{P}$ will intersect the cylinder E_1 in the upper–half space $z > 0$. Let S_1 be the solid bounded by the cylinder E_1, the plane $\mathcal{P}$, and the XY-plane.

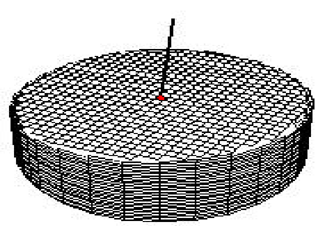

Figure 1.1 The solid S_1 with the normal vector $< s,t,1 >$ at the point $P(0,0,c)$ on its roof

As the parameters s and t change, the roof of the solid S_1 changes. Therefore, the center of gravity $G(\bar{x}, \bar{y}, \bar{z})$ of this solid changes. We would like to study the behavior of this center of gravity G for changing s and t. The coordinates $\bar{x}$, $\bar{y}$ and $\bar{z}$ of G are defined via certain four triple integrals as follows (see [11] and [13]):

$$\bar{x} = I_x / V \tag{1.2}$$

$$\bar{y} = I_y / V \tag{1.3}$$

$$\bar{z} = I_z / V \tag{1.4}$$

In the above equations, the four triple integrals I_x, I_y, I_z and V are defined as

$$I_x = \iiint_D x\,dv\,; \quad I_y = \iiint_D y\,dv\,; \quad I_z = \iiint_D z\,dv\,; \quad V = \iiint_D dv \tag{1.5}$$

where D denotes the solid region defined by the solid S_1. The best way to evaluate the above four integrals is by means of a cylindrical-type coordinate transformation, $x = a\,r\,Cosq$, $y = b\,r\,Sinq$ and $z = z$, where $0 \le r \le 1$ and $0 \le q \le 2p$. The Jacobian J of this transformation is given by the following 3X3 determinant (see [11] and [13]):

$$J = \begin{vmatrix} \partial x / \partial r & \partial x / \partial q & \partial x / \partial z \\ \partial y / \partial r & \partial y / \partial q & \partial y / \partial z \\ \partial z / \partial r & \partial z / \partial q & \partial z / \partial z \end{vmatrix} \tag{1.6}$$

It can be easily verified that $J = a\,b\,r$. Therefore, with the variable transformation, the first of the integrals in equation (1.5) becomes, $I_x = \int_{q=0}^{2p} \int_{r=0}^{1} \int_{z=0}^{c-arsCosq-brtSinq} a\,r\,Cosq\,.abr\;\,dz\,dr\,dq$. The computer algebra system (CAS) *Mathematica* can be used to evaluate this integral conveniently. For example, the following **"Integrate"** command of *Mathematica* can be used to evaluate the above integral (see [10] and [14]):

```
Integrate[a*r*Cos[theta]*a*b*r, {theta, 0, 2Pi}, {r, 0, 1},
                 {z, 0, c-a*r*s*Cos[theta]-b*r*t*Sin[theta]}]
```

The output yields, $I_x = -p\,a^3bs/4$. Similarly, one can obtain that $I_y = -p\,a^3bt/4$, $I_z = p\,ab(4c^2 + s^2a^2 + t^2b^2)/8$, and $V = p\,abc$. Then equations (1.2)-(1.4) imply that $\bar{x} = -a^2s/(4c)$, $\bar{y} = -b^2t/(4c)$, and $\bar{z} = (4c^2 + s^2a^2 + t^2b^2)/(8c)$. This means that the coordinates of the center of gravity of the solid S_1 is given by

$$G = \left(-\frac{a^2s}{4c}, -\frac{b^2t}{4c}, \frac{4c^2 + s^2a^2 + t^2b^2}{8c} \right) \tag{1.7}$$

One can now calculate the locus of G in the three-dimensional space. Using the **“Eliminate”** command of *Mathematica*, or by hand, one can eliminate the two parameters s and t from the three equations $x = -a^2 s/(4c)$, $y = -b^2 t/(4c)$, and $z = (4c^2 + s^2 a^2 + t^2 b^2)/(8c)$. This yields the following equation, which is the locus of the center of gravity G of the solid S_1:

$$z = \frac{c}{2} + 2c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \tag{1.8}$$

Note that the above equation (1.8) represents an elliptic paraboloid (see [11] and [13]).

The preceding discussion leads to the following theorem:

Theorem 1.1 Consider the solid S_1 bounded by (a) the fixed elliptic cylinder E_1 with equation $x^2/a^2 + y^2/b^2 = 1$, where a and b are fixed positive constants (b) the variable plane P through the fixed point P (0, 0, c) $c > 0$, with variable normal vector $< s, t, 1 >$ where s and t are real parameters, and (c) the XY-plane. We assume that the parameters s and t are such that the plane P intersects the cylinder E_1 in the upper-half space $z > 0$. Then the center of gravity G of the solid S_1 is given by $G = (\,-a^2 s/(4c), -b^2 t/(4c), (4c^2 + s^2 a^2 + t^2 b^2)/(8c)\,)$. Furthermore, for changing s and t, the locus of G is an elliptic paraboloid given by the equation $z = c/2 + 2c(x^2/a^2 + y^2/b^2)$.

Proof. For a proof independent of *Mathematica* calculations, refer to [9]. $\square$

The following *Mathematica* program, which utilizes the ideas discussed prior to Theorem 1.1, helps visualize the locus of G in three-dimensional space.

Program 1.1

```
Clear[x,y,z,r,theta,a,b,c]
x=a*r*Cos[theta]; y=b*r*Sin[theta];                    (* Defines the variable transformation *)
j=Simplify[Det[{{D[x,r],D[x,theta],D[x,z]},{D[y,r],D[y,theta],D[y,z]},
                {D[z,r],D[z,theta],D[z,z]}}]];                   (* Calculates the Jacobian *)
ix=Integrate[j*x,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
iy=Integrate[j*y,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
iz=Integrate[j*z,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
i=Integrate[j*1,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
{x0,y0,z0}=Simplify[{ix/i,iy/i,iz/i}]                               (*  Calculates G *)
Clear[x,y,z]
expr=z/.Solve[Eliminate[{x,y,z}=={x0,y0,z0},{s,t}],z][[1]]          (*  Calculates the locus of G *)
p1=ParametricPlot3D[Evaluate[{a*Cos[theta],b*Sin[theta],z} /.
        {a->1,b->2}],{theta,0,3Pi/2},{z,0,10},DisplayFunction->Identity]      (* Plots the cylinder *)
p2=Plot3D[expr/.{a->1,b->2,c->5},{x,-2,2},{y,-2,2},Mesh->False,
               DisplayFunction->Identity]                             (* Plots the locus *)
Show[{p1,p2},PlotRange->{{-2,2},{-2,2},{0,10}}, DisplayFunction->$DisplayFunction]
p3:=Plot3D[c-x*s-y*t/.{a->1,b->2,c->5,t->s/3+Sin[s]+Cos[s]},
        {x,-2,2},{y,-2,2},PlotRange->{0,10},DisplayFunction->Identity]        (* Plots the roof  *)
Do[Show[Graphics3D[ {PointSize[1/40],RGBColor[1,0,0],Point[{x0,y0,z0}]}/. {a->1,b->2,c->5,
   t->s/3+Sin[s]+Cos[s]}], p2,p1,p3,  PlotRange->{0,10},DisplayFunction->$DisplayFunction],{s,-2,2,0.2}]
```

As some outputs of the program, one obtains the coordinates of G, and the equation of the locus of G, as given by equations (1.7) and (1.8) respectively. The program also produces an animation of the center of gravity G. When the animation is run, one can see the different positions of the roof of the solid S_1, along with the graph of the locus of G as an elliptic paraboloid. The position of the center of gravity G can be seen as a red dot, moving along the surface of the elliptic paraboloid inside the cylinder E_1. This is a good way of visualizing Theorem 1.1. A few frames of the animation are given below:

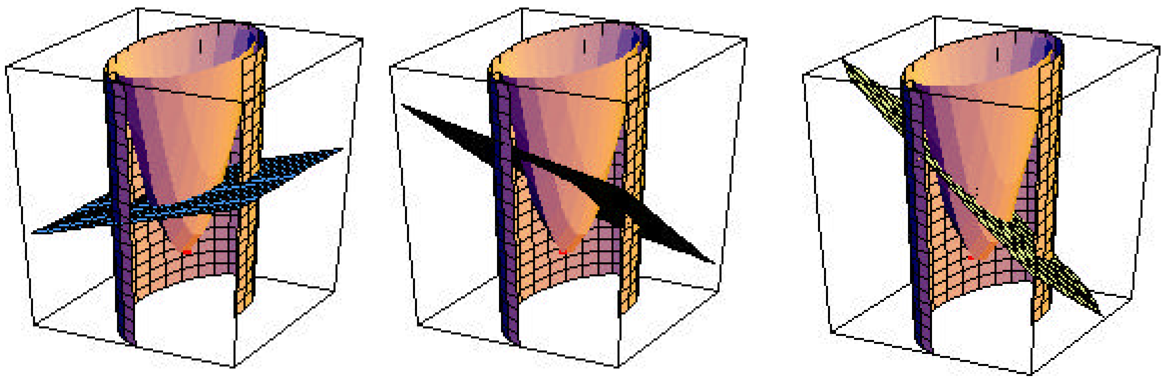

Figure 1.2. An animation of the center of gravity G of the solid S_1

In this section, we also observed how to use the CAS *Mathematica* as a tool for computation and visualization. For general references on *Mathematica*, the reader can refer to [2], [10], and [14]. For the usage of *Mathematica* as a visualization and a conjecture-forming tool refer to [3], [4], [6] and [8]. For the usage of *Mathematica* as an animation tool, the reader can refer to [4], [5] and [7].

In the next section, we will consider a different class of right-cylinders.

2. The Center of Gravity of a Right-Rectangular Cylinder Bounded by a Plane

In the previous section, the base of the solid S_1 was the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. One can now ask the question what will happen if we change the shape of this base. For example, this time one could consider a rectangular base. Let R be the rectangular region defined by the inequalities $-a \le x \le a$ and $-b \le x \le b$, where a and b are fixed positive constants. Let E_2 be the right-cylinder having the rectangular region R as the base, and P be the same plane we had defined in section 1 (see equation (1.1)). We will again assume that the parameters s and t are such that, the plane will intersect the cylinder E_2 in the upper-half space $z > 0$.

Let S_2 be the solid bounded by the right-rectangular cylinder E_2 and the plane P. As before, let $G(\bar{x}, \bar{y}, \bar{z})$ denote the center of gravity of the solid S_2, which are again defined via equations (1.2)-(1.5). In the present case, the triple integrals are performed over the solid region defined by S_2.

However, in order to evaluate the triple integrals, one does not need a variable transformation as in section 1, because the base of the solid is a rectangular region. In order to calculate the coordinates of G, locus of G, and to trace the position of G in the three-dimensional space, one can modify Program 1.1 in the preceding section as given below:

Program 2.1

```
Clear[a,b,c,d,s,t,p1,p2]
box=Graphics3D[{{Thickness[1/80],RGBColor[0,0,1],
  Line[{ {-a,b,0},{-a,-b,0},{a,-b,0},{a,b,0},{-a,b,0}}]}, {Thickness[1/80],RGBColor[0,0,1],Line[{{-a,b,0},
          {-a,b,d}}]},
  {Thickness[1/80],RGBColor[0,0,1],Line[{{-a,-b,0},{-a,-b,d}}]},
  {Thickness[1/80],RGBColor[0,0,1],Line[{{a,b,0},{a,b,d}}]},
  {Thickness[1/80],RGBColor[0,0,1],Line[{{a,-b,0},{a,-b,d}}]}}, Shading->False,Boxed->False]
i1=Integrate[x,{x,-a,a},{y,-b,b},{z,0,c-x*s-y*t}];
i2=Integrate[y,{x,-a,a},{y,-b,b},{z,0,c-x*s-y*t}];
i3=Integrate[z,{x,-a,a},{y,-b,b},{z,0,c-x*s-y*t}];
i4=Integrate[1,{x,-a,a},{y,-b,b},{z,0,c-x*s-y*t}];
{x0,y0,z0}=Simplify[{i1/i4,i2/i4,i3/i4}]
expr=z/.Solve[Eliminate[{x,y,z}=={x0,y0,z0},{s,t}],z][[1]]
a=3;b=2;c=8;d=10;
p1=Plot3D[expr,{x,-a,a},{y,-b,b},DisplayFunction->Identity,Mesh->False]
p2:=Plot3D[c-x*s-y*t/.s->t/3-Sin[t]+Cos[t],{x,-a,a},{y,-b,b},DisplayFunction->Identity]
Do[Show[{  Graphics3D[{PointSize[1/40],RGBColor[1,0,0],Point[{x0,y0,z0}]/.  s->t/3-
Sin[t]+Cos[t]}],box,p1,p2}, DisplayFunction->$DisplayFunction,PlotRange->{{-a,a},{-b,b},{0,d}}],{t,-1,3,0.2}]
```

For general values of a, b and c, the program will calculate the coordinates of the center of gravity G, and the locus of G, given by the following two equations respectively.

$$G = \left(-\frac{a^2 s}{3c}, -\frac{b^2 t}{3c}, \frac{3c^2 + s^2 a^2 + t^2 b^2}{6c} \right) \tag{2.1}$$

$$z = \frac{c}{2} + \frac{3c}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \tag{2.2}$$

According to equation (2.2), the equation of the locus of G is an elliptic paraboloid. It is interesting to compare this with the elliptic paraboloid obtained in the previous section, given by equation (1.8). The program also produces an animation of the center of gravity G, of which a few frames are given below:

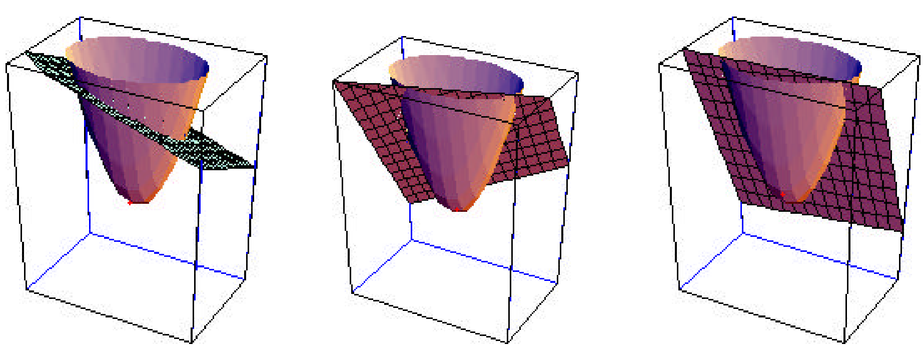

Figure 2.2 An animation of the center of gravity of the solid S_2

Our observations lead to the following theorem:

Theorem 2.1 Let R be the rectangular region defined by the inequalities $-a \le x \le a$ and $-b \le x \le b$, where a and b are fixed positive constants. Let E_2 be the right-cylinder having the rectangular region R as the base, and P be the same plane we had defined in Theorem 1.1. Assume that the parameters s and t are such that, the plane P will intersect the cylinder E_2 in the upper-half space $z > 0$. Let S_2 be the solid bounded by the right rectangular cylinder E_2 and the plane P. Then the center of gravity of the solid S_2 is given by $G = (-a^2 s/(3c), -b^2 t/(3c), (3c^2 + s^2 a^2 + t^2 b^2)/(6c))$. Furthermore, for changing s and t, the locus of G is an elliptic paraboloid given by the equation $z = c/2 + (3c/2)(x^2/a^2 + y^2/b^2)$.

Proof One can directly calculate the four triple integrals corresponding to equation (1.5). For example,

$$I_x = \int_{x=-a}^{a} \int_{y=-b}^{b} \int_{z=0}^{c-xs-yt} x\,dz\,dy\,dx = \int_{x=-a}^{a} \int_{y=-b}^{b} x(c - xs - yt)\,dy\,dx = \int_{x=-a}^{a} x(2bc - 2bsx)\,dx$$

$$= -2bs(2a^3)/3 = -4a^3 bs/3$$

Thus, $I_x = -4a^3 bs/3$. Similarly, one can show that, $I_y = -4ab^3 t/3$, $I_z = 2ab(3c^2 + s^2 a^2 + t^2 b^2)/3$, and $V = 4abc$. Then equations (1.2)-(1.4) imply that $\bar{x} = -a^2 s/(3c)$, $\bar{y} = -b^2 t/(3c)$, and $\bar{z} = (3c^2 + s^2 a^2 + t^2 b^2)/(6c)$, which establishes the first part of the theorem. To calculate the locus of G, eliminate the parameters s and t from the three equations $x = -a^2 s/(3c)$, $y = -b^2 t/(3c)$, and $z = (3c^2 + s^2 a^2 + t^2 b^2)/(6c)$. This yields the relationship $z = c/2 + (3c/2)(x^2/a^2 + y^2/b^2)$, proving the second part of the theorem. □

In the next section, we will explore yet a different class of right-cylinders, known as the generalized right-astroidal cylinders.

3. The Center of Gravity of a Generalized Right-Astroidal Cylinder Bounded by a Plane

Recall that astroid is one of the classical Greek curves, given by the equation $x^{2/3} + y^{2/3} = a^{2/3}$, where a is some positive constant. It has the parametric representation $x = a\,Cos^3 q$, $y = a\,Sin^3 q$, $0 \le q < 2p$. The astroid is indeed the locus of a fixed point on a smaller circle rolling inside a fixed larger circle, where the radius of the smaller circle is one fourth of that of the larger circle (see [5], [7], and [15]). The astroid is a star-shaped closed curve.

One can generalize the equation of the astroid in more than one way. For example, one can consider the plane curve given by the equation $x^{2/n} + y^{2/n} = 1$, where $n > 1$ is an odd integer. Let R be the plane region bounded by this curve, and let E_3 be the right-cylinder having R as the base. We will call the cylinder E_3, a generalized right-astroidal cylinder.

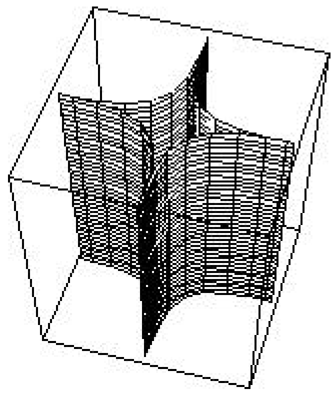

Figure 3.1 The generalized right-astroidal cylinder E_3 for $n = 5$

Let P be the same plane as defined in section 1 (see equation (1.1)), passing through the fixed point $P(0,0,c)$ $c > 0$. We will assume that the parameters s and t are such that the plane P will intersect the cylinder E_3 in the upper-half space $z > 0$. Let S_3 be the solid bounded by the cylinder E_3, and the plane P. We are interested in studying the center of gravity $G(\bar{x}, \bar{y}, \bar{z})$ of this solid S_3, for changing s and t.

Unlike the corresponding situations in the previous two sections, the solid S_3 poses a significant challenge. The main reason is that *Mathematica* shows difficulty in directly evaluating the corresponding integrals in equation (1.5), even with a variable transformation. For example, consider the transformation $x = r\, Cos^n q$, $y = r\, Sin^n q$, $z = z$, where $0 \le r \le 1$ and $0 \le q < 2p$. Using equation (1.6) it is easy to see that the Jacobian J of this transformation is given by $J = n\, r\, Cos^{n-1} q \;\; Sin^{n-1} q$. Therefore, the first integral of equation (1.5) becomes

$$I_x = \int_{r=0}^{1} \int_{q=0}^{2p} \int_{z=0}^{c - s\, r\, Cos^n q - t\, r\, Sin^n q} n r^2 Cos^{2n-1} q \;\; Sin^{n-1} q \; dz\, dq\, dr \tag{3.1}$$

Mathematica shows difficulty in evaluating the above integral (3.1) directly. The strategy is to do the integral (3.1) partly by hand, break into three integrals, and then use *Mathematica*. Without any difficulty, one can perform the innermost integral of equation (3.1) with respect to *z*, to obtain

$$I_x = \int_{r=0}^{1} \int_{q=0}^{2p} n\, r^2\, Cos^{2n-1} q \;\; Sin^{n-1} q \;\; (c - s\, r\, Cos^n q \; - t\, r\, Sin^n q\,)\, dq \;\; dr \tag{3.2}$$

Since n is an odd integer, one can write $n = 2m + 1$ for some integer m. Then it is easy to see that we can break equation (3.2) into the following three integrals:

$$I_x = cn\int_{r=0}^{1} I_{x1}\, r^2 dr - sn\int_{r=0}^{1} I_{x2}\, r^3 dr - tn\int_{r=0}^{1} I_{x3}\, r^3\, dr \tag{3.3}$$

where

$$I_{x1} = \int_{q=0}^{2p} Cos^{4m+1}q\ \ Sin^{2m}q\ \ dq \tag{3.4}$$

$$I_{x2} = \int_{q=0}^{2p} Cos^{6m+2}q\ \ Sin^{2m}q\ \ dq \tag{3.5}$$

and

$$I_{x3} = \int_{q=0}^{2p} Cos^{4m+1}q\ \ Sin^{4m+1}q\ \ dq \tag{3.6}$$

The task is now to handle the integrals (3.4)-(3.6). The first and the last of them are similar, involving an odd power for the cosine term. *Mathematica* again shows difficulty evaluating them directly. However, using a simple trigonometric substitution $u = Cosq$, and the periodicity of the sine function, it is not too hard to establish that $I_{x1} = I_{x3} = 0$. However, *Mathematica* is indeed successful in evaluating the integral I_{x2}, yielding

$$I_{x2} = \frac{2\,\Gamma(m+1/2)\,\mathrm{T}(3m+3/2)}{\mathrm{T}(4m+2)} \tag{3.7}$$

where $\Gamma(x)$ denotes the Euler's Gamma Function (see [1] and [12]). Thus equations (3.3) and (3.7) imply that

$$I_x = \frac{-s(2m+1)\mathrm{T}(m+1/2)\,\mathrm{T}(3m+3/2)}{2\,\mathrm{T}(4m+2)} \tag{3.8}$$

In a similar fashion, one can show that

$$I_y = \frac{-t\,(2m+1)\mathrm{T}(m+1/2)\mathrm{T}(3m+3/2)}{2\,\mathrm{T}(4m+2)} \tag{3.9}$$

$$I_z = \frac{(2m+1)c^2\,4^{-m}\sqrt{p}\ \ \mathrm{T}(m+1/2)}{2\,\mathrm{T}(m+1)} + \frac{(2m+1)\,\mathrm{T}(m+1/2)\,\mathrm{T}(3m+3/2)\,(s^2+t^2)}{4\,\mathrm{T}(4m+2)} \tag{3.10}$$

$$V = \frac{(2m+1)\,c\,4^{-m}\sqrt{p}\ \ \mathrm{T}(m+1/2)}{\mathrm{T}(m+1)} \tag{3.11}$$

Thus, we are able to calculate all four triple integrals I_x, I_y, I_z and V using the Euler's Gamma Function. The Euler's Gamma Function is one of the special functions in mathematics, that is widely used in physics and applied mathematics. For the readers' convenience, we will below include the definition and some properties of this function (see [1] and [12]).

Definition 3.1 For any real number $x > 0$, the Euler's Gamma Function T is defined as $\mathrm{T}(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$.

It can be shown that the improper integral in the above definition is convergent for any $x > 0$. One can use the definition to establish a number of properties of the Euler's Gamma Function, of which a few are listed below (see [1]). These results will be used to derive Corollary 3.1.

Proposition 3.1 The Euler's Gamma Function T has the following properties:

(a) $\mathrm{T}(x+1) = x\,\mathrm{T}(x)$ for any $x > 0$

(b) $\mathrm{T}(n+1) = n!$ for any nonnegative integer n

(c) $\mathrm{T}\left(\frac{1}{2}\right) = \sqrt{p}$

(d) $\mathrm{T}\left(n + \frac{1}{2}\right) = \frac{(2n)!\,\sqrt{p}}{4^n\, n!}$

Proof This follows from the Definition 3.1 above. $\square$

Returning back to the discussion on the center of gravity of the solid S_3, using equations (3.8)-(3.11), one can now calculate the ratios I_x / V, I_y / V, and I_z / V. This yields the following three coordinates of the center of gravity of the solid S_3:

$$\bar{x} = \frac{-2^{2m-1}\, s\, \mathrm{T}(m+1)\,\mathrm{T}(3m+3/2)}{c\sqrt{p}\;\mathrm{T}(4m+2)} \tag{3.12}$$

$$\bar{y} = \frac{-2^{2m-1}\, t\, \mathrm{T}(m+1)\,\mathrm{T}(3m+3/2)}{c\sqrt{p}\;\mathrm{T}(4m+2)} \tag{3.13}$$

$$\bar{z} = \frac{4^m (s^2 + t^2)\,\mathrm{T}(m+1)\,\mathrm{T}(3m+3/2) + 2c^2\sqrt{p}\;\mathrm{T}(4m+2)}{4c\sqrt{p}\;\mathrm{T}(4m+2)} \tag{3.14}$$

One can now use the **"Eliminate"** command of *Mathematica*, or use hand calculations to eliminate the parameters s and t from the above three equations (3.12)-(3.14). This yields the following equation of the locus of the center of gravity of the solid S_3:

$$z = \frac{c}{2} + \frac{c\sqrt{p}}{2^{2m}} \frac{\mathrm{T}(4m+2)}{\mathrm{T}(m+1)\,\mathrm{T}(3m+3/2)} (x^2 + y^2) \tag{3.15}$$

One is now in a position to record the following theorem:

Theorem 3.1 The center of gravity $G(\bar{x}, \bar{y}, \bar{z})$ of the solid S_3 is given by the equations (3.12)-(3.14) above. Furthermore, for changing s and t, the locus of G is an elliptic paraboloid given by the equation (3.15) above.

Proof Most of the proof is contained in the discussion preceding the statement of the theorem. The reader is encouraged to complete the missing details. ∎

Corollary 3.1 The equation of the locus of the center of gravity of the solid S_3 can also be written as

$$z = \frac{c}{2} + c\, 2^{4m+1} \frac{(4m+1)!\,(3m)!}{(6m+1)!\, m!}(x^2 + y^2) \tag{3.16}$$

Proof Follows at once from the equation (3.15) and the Proposition 3.1. ∎

References

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