Aspects of the influence of technology on teaching mathematics in the last decades

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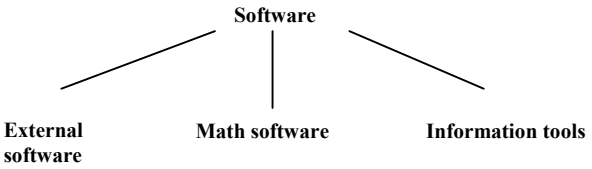
Abstract

The use of technology in teaching and learning mathematics in the last decade substantially influenced teaching and learning mathematics, both its content and practice at undergraduate level. In this paper we present a taxonomy of software created in this process, and will discuss how various mathematical software and some of the internet resources have affected teaching and learning in some of the Australian, British and American universities.

1. Background

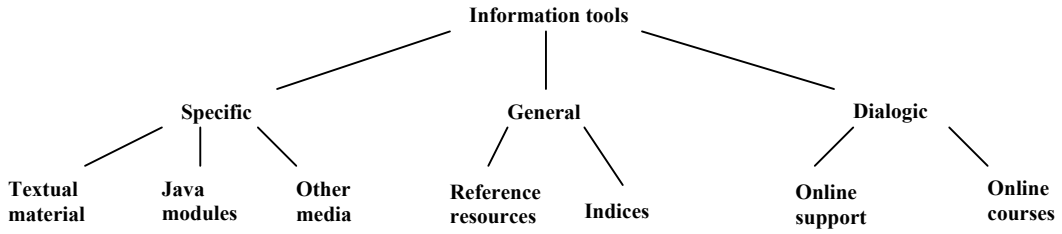
By the terms "the use of technology" we refer to developments such as computer algebra systems, special-purpose mathematical software and multimedia which have influenced the teaching and learning of mathematics in the recent decades. In this paper we will present a taxonomy of the software used for teaching and learning, or specially created for this purpose.

At a general level, we can identify three distinct, possibly overlapping, categories of software:

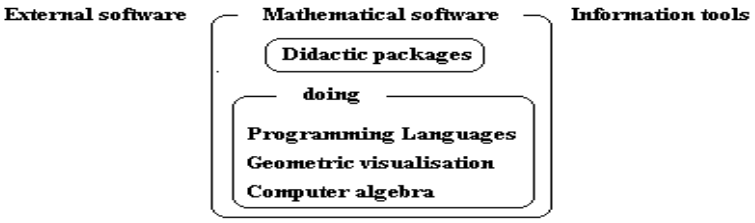


By 'External software' we mean packages which were produced for an entirely different purpose, for example spreadsheets or databases but have been found useful for teaching purposes. There are a number of interesting examples of such use, Sutherland (1994) discusses the issue of getting weaker students to grasp the basics of algebra using the spreadsheet EXCEL.

'Information tools' includes a wide range of initiatives that can be accessed from the World Wide Web. The use of the computer to access remote information is now common to all subjects, and information obtained in this way is rapidly becoming a major educational resource. Moreover this mode of use places the distance student on a more equal footing with the conventional student. We have identified the following sub-categories for the purposes of discussion.



Crowe-Zand (2000) contains a detailed discussion of this part of our taxonomy. The second group 'math software', can be further subdivided. On a broad level there are computer algebra packages (such as Mathematica and Maple) and other *mathematically based* general purpose tools, as well as *didactically based* specific tools such as Tall's Graphic Calculus. Tall (1994) distinguished between the use of the computer for *doing* mathematics and for *learning* mathematics. The former type of software is likely to emulate the logical structure of the subject, and the software for learning mathematics, designed to assist a mathematics undergraduate in learning a specific topic, should take account of the cognitive growth which takes place as the student learns. The position we adopt here is rather different; we consider that any 'mathematical' use of the computer cannot be divorced from learning, so that didactic packages are primarily distinguished by their *lack* of generality for 'doing' mathematics. We shall argue that the sophistication of modern computer algebra packages allows the teacher to accommodate the cognitive growth as advocated by Tall. Therefore we propose to sub-divide the 'math software' category as follows.



Didactic packages are usually tightly tied to a particular topic, and under this heading we include the automatic generation of exercises. The 'doing' packages fall into one of the three remaining types: mathematically oriented programming languages, geometric visualisation tools and computer algebra packages. Notice that (with the exception of didactic tools) all these types may also be used in the amplifier mode (and indeed so may external software). Moreover many computer algebra packages contain user-definable routines, and are thus arguably programming languages. The boundaries in our taxonomy must therefore be regarded as blurred; nevertheless it is useful for organising the discussion. Each subdivision will be dealt with in turn, with representative examples of each topic being presented and discussed.

2. External software

2.1 Spreadsheets

There are many publications dealing with teaching school mathematics using spreadsheets, but perhaps less well known are the developments for the undergraduate curriculum. Recently there have been a number, some using spreadsheets alone and some in combination with computer algebra. Golshan (1997) describes the use of mathematics as a source of interesting applications for a freshman course primarily designed to teach use of Microsoft Excel. Shannon (1994) reports teaching in differential equations which covers numerical solutions. In conjunction with DERIVE (to handle symbolic integration) a complete course on differential equations can be presented. Shelton (1995) has a similar course based on Excel and Mathematica.

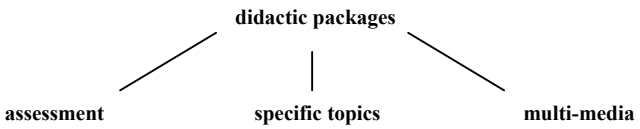
2.2 Theorem provers

There are some interesting new developments arising from the interplay between mathematics and computer science. R. Back has described how the type of proof format which is popular in formal software engineering can be adapted to teach mathematical reasoning. His colleague Joakim von

Wright has shown how the HOL theorem prover can be used in this context (1999). It well known that the role of proof in the mathematics curriculum has diminished significantly over the last two decades, and so any constructive initiative to teach logical thinking should be welcomed.

3 Didactic packages

Under the general heading of didactic package we identify further subdivisions: computer generated assessment (under which we include exercises and self-assessment tests), specific software designed to teach a particular topic, and multi-media software.



3.1 Assessment

There are interesting accounts of the early use of computers to provide a combination of diagnostic testing with carefully targeted remedial tuition (for example Middleton et al. (1989)). Most of the interesting developments in this area arise from universities or colleges. For example, The University of Cincinatti (1996) system for automatic (on-line) generation of questions, and their delivery to students, the 'knowledge-based' diagnostic testing system for basic mathematics, with remedial notes for students, developed under the auspices of UK Teaching and Learning Technology Programme administered by a consortium of universities in the north of England, (1996). At the University of New South Wales teaching of calculus and linear algebra is based on Maple, and this system is used to generate assessment materials. Students can take a test as frequently as they wish without penalty, homework can be submitted online and comments are communicated online.

3.2 Specific didactic packages

Over the last decades, research and development in this area has led to many innovative packages for teaching mathematics. Many packages have been developed by teachers or college instructors, frequently with minimal resources, and for specific purposes such as teaching calculus. Some have been successful. Below we will briefly present some of the current examples.

The Open University

The Open University has produced a third level course on graph theory and combinatorics which makes essential use of a personal computer (Lister (1995)). The tools include a graph editor, a graph database, a "minimum connector" algorithmic package, and a presentation of braced rectangular frameworks. Another didactic package is used for teaching basics of probability and statistics as part of the first year foundation course in mathematics.

The Visual Calculus project (University of Tennessee at Knoxville)

This is a collection of animated notes aimed at explaining some of the difficult aspects of real analysis. An extensive range is available produced by Larry Husch (also a director of the Maths Archives). We do not know of detailed evidence of effectiveness, but the range of topics is impressive.

Tall's computer graphics and calculus

Tall uses computer graphics to teach calculus, (1990). The underlying concept is "local straightness" which states that under sufficiently high magnification differentiable graphs look like a straight line, which allows learner to "see" the gradient of a curve at a point and, Tall argues, this that notion gives an intuitive introduction to calculus.

Teaching Abstract Algebra

John Cannon of the University of Sydney, is one of the first pioneers in developing software for teaching group and ring theory, and presents a history of this contribution.

Two packages designed specifically to facilitate calculations with small groups have been used by Townsley-Kulich (1999) to enable beginning students of group theory to adopt an exploratory approach. Of course there are many other examples (several of which are based on Java) which are available for downloading from the Internet, but these will be described later.

3.3 Multi-media teaching/learning software

The use of computer graphics to aid with the visualisation started over at least two decades ago. Modern developments use sound, computer graphics, pictures and animated images, routinely. This medium, at the current state of technology, suffers the drawback that it is very greedy of resource. In order to make a high quality multi-media teaching package it is essential to have a thorough design process; much programming skill is needed to create a usable environment, and if sound and video are to be incorporated then these should be of equally high quality. The strength of a package is determined by its weakest component, and the implication is that such developments should not be lightly undertaken. We shall consider just three examples, all of which show this level of care.

Mediated Learning

Mediated Learning is a semi-commercial enterprise, originated by Gifford (1995). The textbook metaphor (incorporating multi-media) is used, although the use of the electronic textbook is adapted and amended dynamically in the light of the student's performance. The role of self-assessment is crucial, and feedback is continually provided to learners.

The Cambridge/Keele group

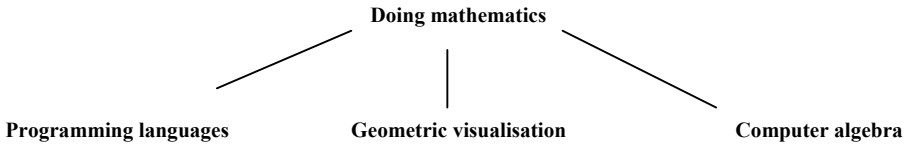
This project has led to the production of several computer-illustrated texts (a combination of book and computer software), in addition two CD-ROMs with a variety of components were produced. Their basic approach was to use multi-media (controlled by the student, in the manner of hyper-card) to teach a variety of mathematical topics to students transferring from school to university.

Open University

The use of multi-media can be particularly effective in the context of mechanics: a recent initiative at the Open University has produced a second year applied mathematics course. Making essential use of a personal computer, the course requires the student to use fifteen multi-media packages based on 6 CD-ROMs. These cover a huge range of applied maths, for example an environment for the exploration of damped and forced vibrations in springs, statics, dynamics (including an introduction to non-linear systems), and heat transfer. Any calculation, graphing or symbolic manipulation that the student needs to perform can be done with a simple switch into a computer algebra package, which is available in the background as a tool.

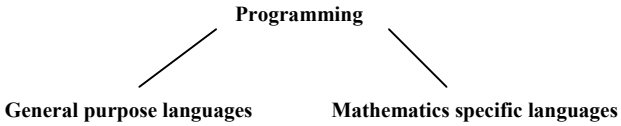
4 Packages for 'doing mathematics'

Under this general heading we have identified three further types of mathematical software, and will give some examples of each.



4.1 Computer programming languages in the mathematics curriculum

Programming has played a part in mathematics degrees for more than thirty years, although initially it was confined to specific parts of the curriculum (such as numerical analysis) using FORTRAN. This is a well known area, and so we will concentrate on the mathematics specific languages.



ISETL

The programming language ISETL is designed to facilitate a large number of mathematical operations and constructions. It is a functional language, though without the strong typing characteristic of ML - in fact it has some similarity with Miranda (Turner (1985)); this has the important consequence that functions are first-class entities and can themselves be operated on by functions. Dubinsky, and his co-workers Schwingendorf and Mathews used ISETL to develop 'the Calculus, Concepts, Computers and Co-operative Learning' (C4L) programme at the University of Purdue. The purpose is to use ISETL to support a constructivist environment for teaching the fundamentals of calculus.

4.2 Geometric visualisation tools

The next part of our taxonomy covers software which is primarily designed to provide visual stimulation. The geometry packages Cabri Géomètre and Geometer's Sketchpad (which originates in America) have aroused considerable interest in universities and schools. In terms of functionality the two are similar, so we confine our attention to Cabri , developed by the French school of mathematics education in Grenoble (Laborde (1993)). The essence of the idea is that figures (easily constructed on the computer screen) are dynamically adjustable, with the computer applying only the essential constraints. Thus the student can identify the invariant properties of a configuration experimentally, and at the same time is freed from the detailed construction of static representations using pencil and paper.

4.3 Computer algebra systems

We come now to one of our chief interests: computer algebra systems. Over the last decade there has been an explosion of interest in this area: indeed the underlying theory of symbolic computation is an active research area for algebraists. At the time of writing there are more than 20 packages: the most widely used are summarised in the following table: almost all are commercial.

Table 1: computer algebra packages

Axiom	Mathcad
Derive	Mathematica
Macaulay	Matlab
Macsyma	Mupad
Magma	Reduce
Maple	Singular

In fact there are several other packages, often developed to meet individual needs, but their impact has been quite small. We shall confine our discussion to those which have been used most widely in the undergraduate curriculum, namely Mathematica and Maple. (Some of the above, e.g. Macaulay, are really research tools.) Both are based on the idea of a 'document', and it is in this format that most courseware has been produced, mostly in calculus, differential equations and 'applicable' mathematics. In part this is due to the universality of calculus in the curriculum and the current difficulties experienced in teaching it. But there is another important reason: the details of calculus are *tractable* using computer algebra!

5. CAS in Australia, the UK and the USA

There are many initiatives and practices in using computer algebra systems (CAS) in countries across the globe. In the following we provide an overview of the initiatives in Australia, UK and USA, where there has been considerable amount of research and development. We shall include first hand reports from a number of universities in these countries visited by the authors during 1998-2000: often the true picture only becomes apparent by meeting the participants. In our interviews with nearly 100 key faculty members in these three countries we found similar histories of using computer algebra systems. Moreover the respective mathematical communities are facing similar problems - socially, academically and financially. Historically, research, development and the use of CAS in these countries have usually started at the grass roots level, by individuals or groups of individuals in various academic institutions. Some of these initiatives then receive financial support from various private and public funding bodies for further development. In the UK, because of the more centralised system of government and higher education, there have been a number of initiatives at national level for introducing technology in teaching and learning.

Problems

There are many problems facing the mathematical communities in these countries, and these have had a retarding effect on the widespread use of CAS. Mathematics, a subject that attracted many able learners until a few years ago, is currently undergoing a crisis of identity in these countries. The number of applicants to study mathematics has sharply declined, mainly due to the uncertainty about the future job market for mathematicians. The financial resources that have traditionally supported mathematical research have also been sharply reduced. This bleak picture, however, is thought to be transient by some faculty members that we interviewed. There are some hopeful signs at least in the US, and to a lesser extent in Australia, that mathematical research should be more supported as a national security imperative, for example through the recommendations in the report of the Senior Assessment Panel of the International Assessment of the US Mathematical Sciences, (http://www.nsf.gov/pubs/1998/nsf9895), and the recent discussion paper published by the Australian Minister for Education, (http://www.deyta.gov.au/highered/otherpub/greenpaper).

As far as the use of technology in teaching and learning is concerned, a common concern among the faculty members interviewed was that the "top-down" encouragement in this direction within the institutions is mainly an administrative cost cutting policy, which is likely to lead to job losses. Disappointingly, there was no strong shared consensus among the members of the various mathematics departments about the educational merits of using technology in teaching and learning mathematics; nor was there a common agreement about the ways mathematics curriculum should be changed as a result of the applications of new technology.

Although there are small scale evaluation results, there is a noticeable absence of reliable, sustained quantitative data about the effectiveness of the use of use of CAS, and this has added to the uncertainty about the effectiveness of their use. One explanation for the lack of evaluation is that those who are involved in research and development is this area are more interested in advancing their technical achievements.

5.1 The Australian position

As part of our research for these three papers we interviewed 37 faculty members in nine universities across Australia. There are many initiatives in using CAS for teaching mathematics to undergraduates, both inside and outside mathematics departments; in engineering CAS are routinely used for teaching. It was generally acknowledged that continued curriculum change was necessary and inevitable, and in many cases it was underway.

There is pressure in Australian universities to offer courses on the Web, but many faculty members view this with suspicion, and consider it as the second best option educationally.

The following table summarises the current use of CAS in a selection of Australian universities.

Table 2: Austalian universities' use of CAS

Monash	Derive Maple
QUT	Maple Matlab
Ballarat	Derive Maple
Melbourne	Maple
Sydney	Magma Mathematica Matlab
UNSW	Maple
RMIT	Maple
U of WA	Matlab Mathematica Maple
UTSydney	Mathematica
U Queensland	Maple Matlab

5.2 The British position

The first serious progress to use computer technology in teaching and learning on a wide scale in the UK was made by the Computers in Teaching Initiative (CTI) which started in 1985 and was funded by the British government. It was a broadly based programme designed to promote the use of new technology across a spectrum of subjects, and some important developments were undertaken in mathematics. Crowe and Zand (2000) discuss the UK situation in more detail.

Current practices in a selection of British Universities

The current state of using CAS in a selection of British universities is summarised in the following table:

Table 3: British universities' use of CAS

Aberystwyth (University of Wales)	Maple
Bangor	Maple
Birmingham	Maple Matlab
Bristol	Maple
Cambridge	Maple
Glasgow Caledonian	Derive
Hull	Mathcad
Imperial	Mathematica Maple
Kingston	Derive Maple
Leeds Metropolitan	Derive
Liverpool John Moores	Derive Matlab
Oxford	Maple
Plymouth	Derive
Queens Belfast	Mathematica
Southampton	Maple
York	Maple

5.3 The position in the United States

In the United States, the National Science Foundation, NSF, has funded much research and development in this area. For example the Instrumentation and Laboratory Improvement programme has provided money to colleges and high schools to buy computer equipment to support their teaching of mathematics. Another example is the Mathematics and Education Reform forum (1998) which counts among its members the universities of California (Santa Barbara), Illinois (Chicago), Michigan, Minnesota, Nebraska, Maryland, Texas (Austin), Washington as well as Rutgers, Oklahoma State and Pennsylvania State. This influential group has set a wide ranging brief to discuss all aspects of the relationship between mathematics and mathematics education. In addition to NSF support, many private foundations, such Sloane Foundation, have generously supported the research and development in this area in the USA over the last decades.

The following table gives a snapshot of the current use of technology in teaching and learning mathematics in a selection of American universities.

Table 4: US universities' use of CAS

University	**Package**
Arizona State	Maple Matlab
Cornell	Maple
Duke	Mathcad, Maple, Mathematica, Matlab
Georgia Tech	Maple
Harvard University	Mathematica
Indiana	Maple
Indiana University of Pennsylvania	Mathematica
Massachusetts Institute of Technology (MIT)	Matlab
Michigan Technological University	Maple Matlab Mathematica
Montana State (CCP)	Mathcad, Maple, Mathematica, Matlab
North Carolina State U	Maple
Old Dominion Uni	Mathcad Maple
Rose Hulman	Mathematica
San Francisco State University	Mathematica
Southwestern University	Mathematica
St Louis Uni	Maple
Stanford University	Matlab
Texas A&M University	Maple
Uni of N Carolina	Mathcad
University of California, Los Angeles (UCLA)	Maple
University of Carolina Wilmington (UNCW)	Maple Mathcad
University of Colorado at Boulder	Matlab Maple Mathematica
University of Hawaii	Derive
University of Illinois	Mathematica
University of Michigan	Maple Matlab
University of New Hampshire	Matlab Maple
University of Pittsburgh	Mathematica
University of Vermont	Mathematica
University of Virginia	Maple
University of Washington	Matlab Mathematica Maple
Yale	Mathematica

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