TEACHING ELEMENTARY CALCULUS WITH CAS CALCULATORS

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Abstract

In this paper we present our experiences of introducing the CAS graphics calculator into a first year elementary calculus unit taught recently at Edith Cowan University in Perth. We used a Reform Calculus textbook and the fourfold analytic, geometric, numeric and word-description approach advocated in it. The students comprised a mixture of mature age students and school leavers, and they had a wide variety of mathematical and (non-CAS) calculator backgrounds. They were studying calculus as a service unit for a number of different major programs. A set of HP49G CAS graphics calculators was available for student use in class, in the tests and in the final examination. Some HP49G calculators were available in the library for out of class use. The essential difference between using standard graphics calculators for teaching/learning in the unit and using the CAS calculators is the ability to hand analytic procedures to the higher level technology, and associated with this are some new possibilities for learning. We give an overview of the approaches we took to teaching and then assessing with these calculators; and, in particular, we describe the approaches we took with limits, the problems encountered and our responses to the outcomes. We examine also aspects of the CAS use revealed by an analysis of selected test and assignment questions.

1. Introduction

The mathematics teaching staff at Edith Cowan University (ECU) have long practised the philosophy that modern technology should, wherever possible, be integrated into the teaching and learning programme. See, for example, [1]-[3]. This has included the use of graphics calculators without a computer algebra system (CAS) and CAS computer software packages such as *Scientific Notebook*, in a number of our Units. In addition, graphics calculators without CAS have been used extensively in Western Australian secondary schools for a number of years and have been assumed in the Year 12 Tertiary Entrance Examinations (TEE) since 1998. Thus, school leavers entering mathematics units at ECU can be expected to already own and be able to use such a graphics calculator. However, this is not the case with the mature-age students. In the Year 2000 the decision was made to embrace in the unit *MAT1136 Elementary Calculus* the *Reform Calculus* approach as encapsulated in the book written by Hughes-Hallett et al [4], which was chosen as the unit textbook. This approach involves not only the use of technology but the systematic integration into the teaching and learning of viewing each topic from as many as possible of the analytic, geometric, numeric and word-description aspects. The implementation in 2000 was not particularly successful and a number of improvements for the 2001 implementation were noted [2]. These included the introduction of more interactive sessions at which the students work (using their calculators) on a range of selected examples/investigations, and the running of a graphics calculator

workshop at the start of the teaching semester. In addition, it was decided in 2001 to incorporate the use of a CAS graphics calculator into this unit. The implementation and outcomes for 2001 will be detailed here, with particular emphasis on the use of the CAS calculator in the teaching and learning of *Limits*.

2.   Implementation

In first semester 2001 fifty-nine students from a wide range of tertiary program areas completed *MAT1136 Elementary Calculus*. The corresponding Year 2000 number was fifty-eight. The unit was offered on two campuses (ML and JO with thirty and twenty-nine students respectively) and had a mix of school leavers and mature-age students. Class contact time comprised two two-hour teaching sessions per week for twelve semester weeks, together with extra sessions for sitting tests. The mathematical backgrounds of the students varied from some initial calculus exposure at high school to extremely weak precalculus skills. Graphics calculator experience also varied. An entry survey on prior calculus background and graphics calculator familiarity gave the results shown in Table 1, with the corresponding Year 2000 figures in brackets.

Table 1: 2001 Calculator / Calculus background (2000 figures in brackets)

	ML	JO	ALL
Calculator	63% (46%)	79% (74%)	71% (55%)
Calculus	70% (59%)	83% (84%)	76% (67%)

For this unit students were required to have their own CAS or non-CAS graphics calculator. In addition, a set of HP49G CAS graphics calculators was available on loan from Hewlett-Packard for student use in class, in the tests and in the final examination. Two HP49G calculators were available in each campus library for out of class use. Although we recognised that the lack of full-time access to the CAS calculators was problematic (other than for the few students who purchased them), we did not have the resources to provide each student with the technology for his/her personal use.

The unit was taught by Pat (second author). She was teaching the unit for the first time and was initially unfamiliar with the CAS technology. However, she is an experienced upper-secondary mathematics teacher and had prior experience of teaching calculus where all her students owned a lower-level (non-CAS) HP38G graphics calculator. Lyn (first author) and Ute (third author) had taught the unit previously and were involved in the unit planning. Lyn was the Unit Coordinator and assisted with the initial graphics calculator workshop (which centred on use of the HP49G calculator) and initial technology-based investigations, and oversaw the unit assessment. Pat approached all the class sessions as 'interactive tutorial' sessions, rather than formally dividing them into lecture and tutorial sessions. Students were expected to work through structured examples and encouraged to draw out/investigate ideas/concepts themselves, perhaps in discussion with each other (by their choice), and then to contribute to whole-group discussion. Visual methods were encouraged, in keeping with the conceptual emphasis of the prescribed text. There was initial resistance from the students to the mode of delivery and to the emphasis on concepts rather than procedures, particularly at JO where there were more recent school-leavers than in the group at ML. Another factor was that the lecture-theatre layout at JO, with superfluous capacity that allowed students to sit in isolation, was less conducive to student interaction than the classroom one at ML.

The unit assessment comprised three tests (each 45 minutes long and worth 10%), an assignment (worth 20%) and a final examination (worth 50%). In order to give the students with a weak calculus background a chance to 'catch up', and to encourage mastery of the material itself, each

student was required to take one test but was permitted to take two (different) tests on each section tested. A student who scored 80% or higher on any attempt was credited with the full 10% for that test. A student who did not reach this level on either attempt was credited with the higher of his/her two results. Table 2 gives details of the overall pass rate for the unit (with the corresponding Year 2000 figures in brackets), indicating improvement for the current implementation over the previous year, even allowing for the better backgrounds in 2001 of the ML students. However, because of the issues of what comprises an appropriate assessment item and allocation of marks for elementary calculus in a CAS environment, we are cautious in claiming the high success that the figures could be taken to imply. In particular, the balance between routine (procedural) questions and higher-level questions (testing conceptual understanding) needs to be redefined upon introduction of the CAS. We changed the balance between the types of questions in the various tests and examination, but this is an area we will reconsider for next year.

Table 2: Pass rate for 2001 (Year 2000 figures in brackets)

	ML	JO	ALL
Passed Unit	90% (62%)	86% (63%)	88% (62%)

3. CAS Calculator Implementation Difficulties

At the beginning of the unit, students were introduced to the HP49G CAS calculator via an introductory workshop and this was supported by a handout [5] defining basic numeric and graphing capabilities for the workshop, and CAS functions for reference later in the course. This material was a direct translation to the HP49G of our material previously written for the HP48G non-CAS graphics calculator [6]. The workshop booklet was also one resource from which Pat first learnt the functionality of the calculator. Her working through the instructions in it before giving it to the students served to check that the recommended key-stroke sequences gave the expected outputs. However, later, when attempting different examples, it became evident that some of the sequences were not widely generalisable. The problems encountered were a setback to the CAS implementation. Eventually though, through press and play, discussions among ourselves, and consultation of the manuals, most of the problems were resolved. In particular, the roles of the four cursor keys in entry and editing were identified, so that these tasks were simplified. The functionality of the cursor keys on the HP49G is different, and more complex, than those on the lower-level graphics calculators. The same can be said of the graphing platform, following as it does the HP48G, rather than the HP38G approach; and we note that, because of this design feature, transition to using an HP49G is potentially easier for HP48G users than for HP38G users. Another aspect of the HP49G calculator is its range of capabilities, which is a strength, but also a weakness in that their presence complicates navigation of the tool.

4. Teaching approach and outcomes for *Limits* with the CAS calculator

The unit started with the development of the concepts of a real valued function and families of functions (including linear, exponential, logarithmic, trigonometric, polynomial and rational functions). Early on, the ideas of limiting values and functions increasing without bound were introduced informally, in relation to:
- the behaviour of graphs near plus or minus infinity;
- singularities of simple rational functions.

Later:

- approaches to the evaluation of limits using analytic methods and tables of values were addressed extensively, and linked to features on graphs of the corresponding functions;
- evaluation using the CAS capabilities was included, as a check on other methods and also as a first option.

One of the vehicles for the development of the limit concept and for the establishment of different evaluation methods was a self-paced, investigative task on trigonometric limits. Students were asked to deduce the limits as x tends to zero for $\dfrac{\sin x}{x}$, $\dfrac{\sin 2x}{2x}$, $\dfrac{\sin 3x}{2x}$ and $\dfrac{\sin(1/x)}{(1/x)}$. For each question, students were asked to:

- graph the denominator and numerator (see Figure 1a for the $(\sin x)/x$ example), and to deduce and explain the value of the quotient near zero;
- tabulate values of the denominator and numerator near and at zero (see Figure 1b), and divide them, in order to establish (again) the limiting value of the quotient;
- graph the quotient and explore its values near zero using the trace facility on the calculator--the HP49G does not register a function value at $x = 0$ for the given examples (e.g., see Figure 1c), while 'Undefined' shows on some other calculators;
- check the limit of the quotient with the CAS.

A last part was the evaluation of $\displaystyle\lim_{x\to\pi/2}\dfrac{\sin(\pi/2 - x)}{x - \pi/2}$ using the same approaches. A common mistake with the tabular (numerical) approach was dividing zero by zero to yield one, which was addressed with students individually. We did not systematically record outcomes of the task, but it seemed successful in getting the students to think about limits, to the extent that we would reuse it.
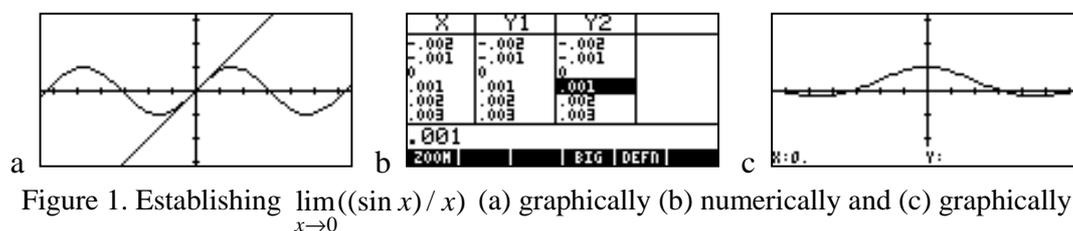

Figure 1. Establishing $\displaystyle\lim_{x\to 0}((\sin x)/x)$ (a) graphically (b) numerically and (c) graphically.

Also, with the work on limits, the limiting values at $\pm\infty$ were returned to, this time for more complex rational functions and the function $Q = S(1 - a^t)$, where these were linked (as had been done before) to the presence of horizontal asymptotes on the corresponding graph. That this link is made explicit is important as students commonly do not recognise horizontal asymptotes on calculator-generated graphs [7, 8] and because it assists the choice of an appropriate vertical scale for graphing on the calculator. Thus the development of the limit concept was very much centred on implications for graphing with the calculator.

Several problematic aspects of using the HP49G CAS capabilities became evident through the work on limits. First, with limits that yield a numerical value (e.g., see Figures 2a and 2b), the processes of simplification are not all available even with the 'step-by-step' calculation option selected. Consequently it is not always apparent when substitution will yield 0/0. A result of 0/0 signals a discontinuity. Point discontinuities, in particular, are not always evident on calculator-generated graphs [7], so students need to be able to predict them. Therefore, with the evaluation of a limit, substitution first is recommended, especially when graphing is to follow. The result of the substitution is available in the table of values in a calculator, which allows identification of a discontinuity through 'undefined' showing in place of the value of the function (see Figure 2c).

However, a problem here is the decimal output if exact values for a limit are required. One strength of the CAS functionality is that it will produce exact values (see Figure 2b).
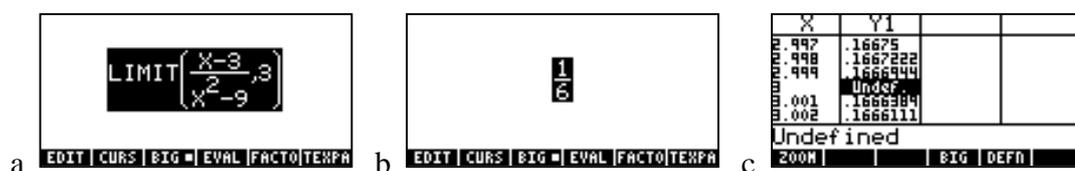

Figure 2. Evaluation of $\lim\limits_{x \to 3} \dfrac{x-3}{x^2-9}$.

Another problem with using the HP49G CAS is that it cannot adequately handle either one-sided limits or singularities. For example, $\lim\limits_{x \to 1} \sqrt{x-1}$, which does not exist, is returned with a value of 0 (which is the right limit obtained directly by substitution). Also, where substitution would yield a constant divided by zero, for example $\lim\limits_{x \to 3} \dfrac{x+3}{x^2-9}$, the CAS output is ∞ (see Figure 3a and 3b) as opposed to a more reasonable flag indicating the behaviours at the singularity. In this case an output of $\pm\infty$ would be a more appropriate choice, as it correctly indicates that the function values differ in sign on either side of $x = 3$. These limitations of the CAS are potentially confusing and misleading in early development of the limit concept.

Figure 3. CAS evaluation of $\lim\limits_{x \to 3} \dfrac{x+3}{x^2-9}$.

As to identified outcomes, the extent to which students used the different approaches for evaluating limits was recorded for a test question. The example below gives the JO version; the corresponding function for the ML group was $f(x) = (5^x - 1)/x$. We note that the question, in keeping with reform calculus, allows students to choose the mode of representation for the evaluation of the limit (numeric/tabular, graphical, 'hand' algebraic, CAS) and calls on word descriptions.

Example 1

a)   Evaluate $\lim\limits_{x \to 0}(4^x - 1)/x$ correct to two decimal places and explain briefly how you obtained it.

b)   Check your answer for a) using a different approach, and explain briefly how you did it.

c)   Sketch the graph of $f(x) = (4^x - 1)/x$ over the domain [-2, 2].

The results are summarised in Table 3. Some students used L'Hopital's rule but, at the time of the test, this had not been encountered in the unit.

Table 3. Extent to which different approaches were used for evaluating limits.

	Table of values	Graph	CAS	L'Hopital's rule
ML for 2a	18	1	1	
ML for 2b	1	2	6	1
JO for 2a	16	1	2	1
JO for 2b	3	6	2	1

[a]Number of candidates for the test were ML 26, JO 29

Furthermore, the incidence of inclusion of the point discontinuity in part c) was ML 14 and JO 15. The relatively small number of students who used the CAS is explained by the emphasis in class that it hides the 0/0 result and that this is important for graphing, which was called for in part c). Yet, seven students in the ML group and eight students in the JO group who used a table of values for the limit still proceeded to omit the discontinuity, indicating superficial understanding of the limit concept. Another possible reason for students not using the CAS is the difficulty and confusion experienced with operation of the calculator (see Section 3).

Another recorded outcome is students' scores for an assignment question (see Table 4), where all students who submitted the assignment attempted the question. It was specifically designed to test if students understood the implications of limits for graphs and again students could choose which representation to use for the solution, and word descriptions were required.

Example 2

Write the value of $\lim_{x \to 2} f(x)$ *for the following functions and explain the implications of your answers for the graph of* $f(x)$ *for each part.*

a) $f(x) = \dfrac{x-2}{x^2+4}$, b) $f(x) = \dfrac{x-2}{x^2-4}$, c) $f(x) = \dfrac{x+2}{x^2-4}$

Table 4. Results on an assignment question on limits

[a]Score	1	2	3	4	5	6
Frequency (ML)	1	3	10	5	4	5
Frequency (JO)	-	2	10	7	4	5

[a] maximum = 6

Being an assignment, students choosing to use a graphics calculator would most likely have relied on their own, non-CAS ones, in preference to accessing the CAS calculators in the library. A score of 3 was generally attributable to correct evaluation of the three limits, without making the correct links to the graphs. So, again results indicated that a significant number of students had low-level understanding of the limit concept.

Limits in the context of graphing were revisited, but then the definition of the derivative as a limit needed to be discussed, when there were still substantial input and editing problems with the CAS. But it seemed that simplification of the difference quotients that are typically encountered was where the CAS could be particularly useful. Such simplifications are often complex and, as such, potentially distract rather than help students understand the definition of the derivative. Thus students were asked, again via a collaborative activity, to illustrate on a diagram, and set up and evaluate with the CAS, the derivatives for a number of polynomials, for example, $f(x) = x^5$ (see Figure 4) and then for other functions, including trigonometric and logarithmic functions. Derivatives at x, and for particular values of x were included. Step-by-step calculation could have shown the intermediary simplifications (e.g., for the limit shown in Figure 4), but students were not

directed towards this option because of possible distraction in coming to understand the derivative concept.

Figure 4. Screen displays of the derivative of $f(x) = x^5$ as a limit.

The CAS syntax for differentiation was also introduced at this stage and served as a check on the limit evaluations (i.e., served to check students had set up the limits correctly). Those who had some calculus background (see Table 1) knew the procedures for polynomials but the derivatives of other functions seemed unfamiliar to most of them.

Again a number of problems/unexpected outcomes were encountered. The first was associated with the priority the calculator attaches to the variable x (see Figure 5), which has also been noted in [9]. So, with a limit evaluation, if variables other than x are involved, they need to be specified. For example, for the expression in Figure 5b, '0' needs to be preceded by 'H=', as in Figure 4.
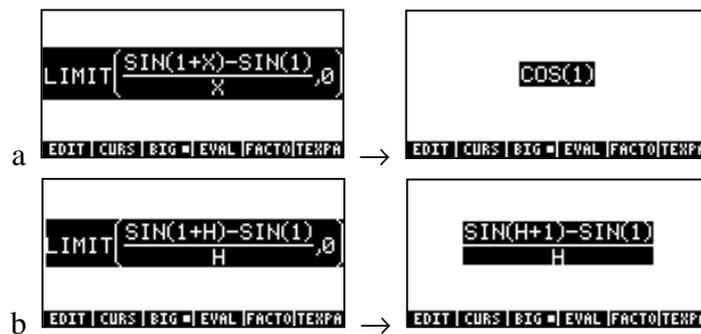

Figure 5. The derivative as a limit for $f(x) = \sin x$ at $x = 1$, illustrating the privileging of x.

Secondly, the CAS gives non-standard outputs for the limits used to obtain the derivatives of some functions. Use of these was avoided in the teaching. They included $f(x) = \tan x$ and $f(x) = a^x$ (see Figure 6). An added complication with these two functions is that the derivative function, which students were using to check their limit derivations, yields outputs that are in a different form from that obtained via the limit method (see Figures 7).
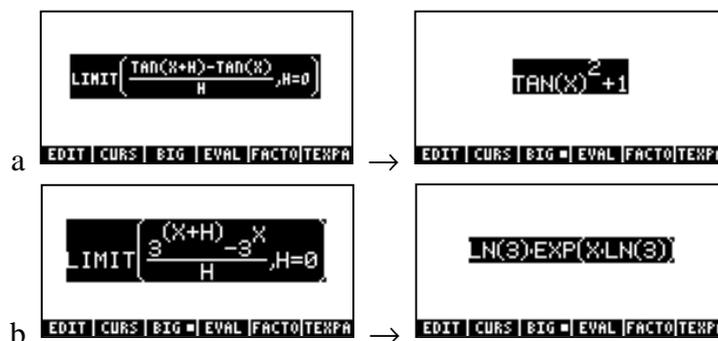

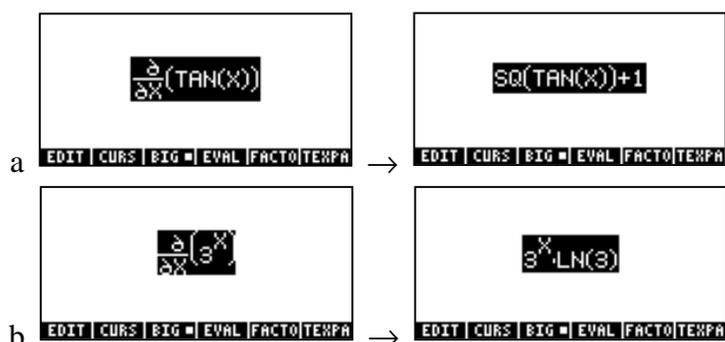Figure 6. The derivative of (a) $\tan x$ and (b) 3^x as a limit.

Figure 7. The derivative of (a) $\tan x$ and (b) 3^x using a CAS derivative function.

Other aspects of the development of derivatives were, as per the textbook, the relation of the concept to practical situations and to tables of function values, and also the construction of the graph of the derivative from the graph of the function. Doing this for $f(x) = \sin x$ included calling on the results from the limit activity described above (under Figure 1), in particular that the gradient for both $\sin x$ and x at $x = 0$ was one. So $(0, 1)$ was used to start the derivative graph and symmetry yielded the points $(\pi, -1)$, $(2\pi, 1)$ etc. This was one of a number of connections that emerged in teaching with a strong orientation towards visual methods. Then, the points on $f'(x)$ corresponding to the turning points on $\sin x$ were plotted, giving more form to the $f'(x) = \cos x$ result.

The algebraic procedures for evaluating derivatives of various functions were practiced along with the development of the derivative concept. One aspect of this was helping students to become aware when calculation using the CAS was an efficient option. 'Simple' functions can be calculated by hand more quickly than the time taken to key them into the CAS, and there is also the issue of interpretation of the output. An indication of the extent to which students relied on 'hand' calculation and the technology, and problems encountered, is indicated by the outcomes on the examination question:

Example 3
Calculate the first derivative for each of the following.

a) $y = 6^x + x^6$ b) $f(x) = 5\ln(x^4 - x)$ c) $f(t) = 5e^{2t}\tan 3t$ d) $h(x) = \dfrac{\cos 3x}{x^2 + 7}$

The extent of calculator use (deduced from no working being shown and by the form of the answer) and student performance on the question is summarised in Table 5.

Table 5: Calculator use for Differentiation (average marks in brackets, total marks were 6)

	ML	JO	ALL
Used calculator in some or all parts	43% (4.8)	50% (5.1)	47% (5.0)
Didn't appear to use the calculator	57% (4.9)	50% (4.3)	53% (4.7)

These results show that students, whether by calculator or not, have coped well with these questions and also reflect the finding [10] that use of the CAS is not the favoured choice. For those students who appeared to use the CAS, marks were lost mainly in part (c) because:
- the answer provided was in terms of x rather than t, or with a mixture of t and x. This is attributable to the student using the DERVX function on the HP49G which requires x as the

variable, or using x with other available derivative functions (e.g., see Figure 7) (where variables other than x can be used but this introduces the tedium of more key strokes);

- an answer of zero was given. This is attributable to the student instructing the calculator to differentiate in terms of x (e.g., by using the DERIVX function) but inputting the function in terms of t;

- incomplete answers. This is attributable to part of the answer being off the screen and students not recognising this or not knowing how to cope with it--intuitive use of the 'right-hand' cursor to move across the screen does not work;

- $\tan{(3t)}^2$ or $\sqrt{\tan 3t}$ was written instead of $\tan^2 3t$ in the answer. The errors are due to misinterpretation of the ambiguous $\tan{(3t)}^2$ and SQ(TAN(X)) calculator syntax (see Figure 7). The calculator output is $\tan((3x)^2)$ if the $(3x)$ is to be squared.

5.   Concluding Discussion

Upon the introduction of the CAS for the teaching and learning of limits students could pass calculation of limits to the calculator, but where the evaluation of limits was to support graphing, substitution was recommended first. In addition, when incorporating use of the CAS into examples for the early development of the limit concept, limitations of the calculator needed to be taken into account. The CAS supported also the development of the derivative concept, but a problem is non-standard outputs for the derivatives of some functions, and interpretation of syntax.   Another issue with the HP49G calculators is the difficulty of transferring from the lower-level technologies (the HP38G rather than the HP48G), especially for graphing.

With increased emphasis on visual approaches, which are consistent with the reform calculus approach, there needs to be clear articulation of skills of visual interpretation. These are rarely alluded to in textbooks and are an area for further research attention. In addition, introduction of the CAS calculator necessitates that assessment is rethought. Replacement of traditional questions testing procedures with application questions or questions calling on graphing and other capabilities of the calculators raises the overall level of difficulty of a test or examination paper. This can be demoralising and might influence students to abandon the subject. There is also a potential problem in the next unit for students who have relied extensively on a CAS calculator for derivative and integral calculations but who do not own a calculator nor have a wide repertoire in what they can carry out 'by hand'.

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