An interactive graphical system
for cam motion synthesis

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Abstract

In this paper we describe the mathematical capabilities of a "special" interactive system with graphical facilities for solving the problem of designing the motion law of cam-follower mechanisms.

Keywords: computer graphics, cam synthesis, approximation , regularization.

1 Introduction

A "special" graphical system for interactive work in the field of "computer aided cam design" (an industrial application of Mechanics) has been found to be useful and highly desirable. A cam is a mechanical device which by its rotation causes a follower to take different positions. The follower displacement may be expressed as a continuous function of time or, more conveniently, of "angular displacement". The range of this function is a curve which is called the motion law of the cam-follower system. Because of the dynamical effects, it is required that the rise and return of the follower are as regular as possible; indeed, a bad choice of the motion law will result in vibrations, wear, noise and even breakdown. In practice, it is therefore necessary that the function that describes the motion law is continuous and has a continuous first and

second derivative.
The so-called classical motion laws (harmonic, trapezoidal, cycloidal, etc.) fulfil the above demands of regularity for the curve, but, in many applications, the requested course of the follower is recorded experimentally and then none of these classical laws can be used to fit the tabulated data accurately. Besides, in the phase of modelling the cam motion law, the designer must refine the expressions for displacement that he obtains to reduce an acceleration peak or to shift peak values of some kinematic parameters away from critical regions.
It is well known that the effectiveness of a "special" system is dependent on the range of the mathematical capabilities of the system.
The use of spline functions to define the motion law of the cam-followers system yields a general, systematic, reliable approach to the task. (e.g. [1], [6], [8].) Also the Bezier computer graphics technique to synthetize follower motion laws has been introduced [7]).
Even if the problem of designing the motion law has been regarded as a higher order smoothing approximation problem, the standard software packages on curve fitting (see, for example [4]) are not readily adaptable for accomodating the constraints on follower displacements, velocities, and accelerations, especially when the set of constraints becomes large.

In this paper we describe the mathematical capabilities of an interactive system with graphical facilities which is based on a very general formulation of the problem of designing the motion law of a cam-follower system.

2 Problem statment

If y represents the displacement and x represents the angular displacement, varying from $a \geq 0$ to $b > a$, it is required that the motion law is expressed by an equation $y = f(x)$ for $a \leq x \leq b$, where $f(x)$ is a function of class $C^2[a, b]$, at least. This allows the first, second and third derivatives of $f(x)$ to be considered as non-dimensional geometric velocity, acceleration and jerk. Besides, it is necessary that the acceleration peak values are reduced as much as possible and are shifted away from critical regions.

Normally the values $f(x_1), f(x_2), \cdots, f(x_m)$ of the function $f(x)$ are assigned at m distinct points $x_1, x_2, \cdots, x_m$ of the interval $[a,b]$; at some of these points also the values of the first and second derivatives are known.

Given a set of data values $\{x_r, f(x_r), f^{'}(x_r), f^{"}(x_r)\}$ with $a \leq x_r < x_{r+1} \leq b$ and a corresponding set of nonnegative weights $\{w_r,\ w_r^{'},\ w_r^{"}\}$, $r = 1, 2, \cdots, m$, we want to determine a spline function $\varphi(x)$ on $[a,b]$, of specified degree $k \geq 5$, with knots $a < \xi_1 < \xi_2 < \cdots < \xi_g < b$, as the solution of the optimization problem:

minimize

$$\left\{\gamma J(\varphi) + \sum_{r=1}^{m} w_r(\varphi(x_r) - f_r)^2 + \sum_{r=1}^{m} w_r^{'}(\varphi^{'}(x_r) - f_r^{'})^2 + \sum_{r=1}^{m} w_r^{"}(\varphi^{"}(x_r) - f_r^{"})^2\right\} \tag{1}$$

where $f_r \equiv f(x_r)$, $f_r^{'} \equiv f^{'}(x_r)$, $f_r^{"} \equiv f^{"}(x_r)$ for $r = 1, 2, \cdots, m$ and $J(\varphi)$ is the functional:

$$J(\varphi) = \int_{\tilde{a}}^{\tilde{b}} (\varphi^{"}(x))^2 dx \tag{2}$$

with $[\tilde{a}, \tilde{b}] \supset [a,b]$, which is a measure of the regularity of the curve $y = \varphi(x)$. The positive parameter γ, called "regularization parameter", controls the extent to which the defect from the data and the smoothness of the curve will be satisfied.
It is often required to control the zero end-point derivative constraints of the cam-follower system. In this case it is convenient to replace the functional (2) into

$$\tilde{J}(\varphi) = J(\varphi) + \beta\left\{\sum_{j=0}^{3} wa_j \mid \varphi^{(j)}(a) \mid^2 + \sum_{j=0}^{3} wb_j \mid \varphi^{(j)}(b) \mid^2\right\} \tag{3}$$

where $\beta \geq 0$ and wa_j, wb_j, $j = 0, 1, 2, 3$, are non negative weights.

3 Representation of the solution

Let $S_k(\xi_1, \xi_2, \cdots, \xi_g)$ denote the vector space (of dimension $g+k+1$) of spline functions on $[a, b]$ of degree $k > 0$ having as knots ξ_i the strictly increasing sequence $a < \xi_1 < \xi_2 < \cdots < \xi_{g-1} < \xi_g < b$. By introducing additional knots $\xi_{-k}, \cdots, \xi_0 = a$ and $\xi_{g+1} = b, \cdots, \xi_{g+k+1}$ satisfying

$$\tilde{a} = \xi_{-k} < \cdots < \xi_{-1} < \xi_0 = a$$

$$b = \xi_{g+1} < \xi_{g+2} < \cdots < \xi_{g+k} < \xi_{g+k+1} = \tilde{b}$$

let $\{B_{k,i}(x)\}$, $i = -k, -k+1, \cdots, g$, denote a basis for $S_k(\xi_1, \xi_2, \cdots, \xi_g)$ of normalized B-spline functions of degree k with support $[\xi_i, \xi_{i+k+1}]$. This basis for S_k has the interesting property of forming a partition of unity on $[a, b]$, i. e.

$$\sum_{i=-k}^{g} B_{i,k}(x) = 1 \qquad \text{for all } x \in [a, b].$$

For any $x \in [a, b]$, the normalized B-splines that are nonzero in the subinterval containing x may be computed by recurrence ($k = 1, 2, \cdots$):

$$B_{k,i}(x) = \frac{x - \xi_i}{\xi_{i+k} - \xi_i} B_{k-1,i}(x) + \frac{\xi_{i+k+1} - x}{\xi_{i+k+1} - \xi_{i+1}} B_{k-1,i+1}(x)$$

$$B_{0,i}(x) = \begin{cases} 1 & if \quad x \in [\xi_i, \xi_{i+1}] \\ 0 & if \quad x \in [\xi_i, \xi_{i+1}] \end{cases}$$

Every spline function $\varphi(x) \in S_k(\xi_1, \xi_2, \cdots, \xi_g)$ has a unique representation

$$\varphi(x) = \sum_{i=-k}^{g} c_i B_{k,i}(x) \tag{4}$$

in which the c_i are called "B-spline coefficients" of $\varphi(x)$.

Substituting the expression for $\varphi(x)$ in (1) and (2) and differentiating with respect to c_i, we obtain the system of equations for the determination of the B-spline coefficients c_i in the form:

$$(\gamma D + E^T W E)\underline{c} = E^T W \underline{h} \tag{5}$$

where $\underline{c} = (c_{-k}, c_{-k+1}, \cdots, c_g)^T$, $W = diag\{\tilde{w}_i\}$, $i = 1, 2, \cdots, 3m$ with

$$\widetilde{w_i} = \begin{cases} w_i & \text{for } i = 1, 2, \cdots, m \\ w'_{i-m} & \text{for } i = m+1, m+2, \cdots, 2m \\ w''_{i-2m} & \text{for } i = 2m+1, 2m+2, \cdots, 3m \end{cases}$$

$\underline{h} = (f_1, f_2, \cdots, f_m, f'_1, f'_2, \cdots, f'_m, f''_1, f''_2, \cdots, f''_m)^T$, E is the rectangular matrix $(3m \times (g+k+1))$ with the (i,j)-th element

$$e_{i,j} = \begin{cases} B_{k,j}(x) & \text{for } i = 1, 2, \cdots, m \\ B'_{k,j}(x_{i-m}) & \text{for } i = m+1, m+2, \cdots, 2m \\ B''_{k,j}(x_{i-2m}) & \text{for } i = 2m+1, 2m+2, \cdots, 3m \end{cases}$$

$(i = 1, 2, \cdots, 3m;\ j = -k, -k+1, \cdots, g)$, and D is the square matrix $(g + k + 1) \times (g + k + 1)$ with the (i,j)-th element

$$d_{i,j} = \int_{\tilde{a}}^{\tilde{b}} B''_{k,i}(x) B''_{k,j}(x) dx$$

$(i, j = -k, -k+1, \cdots, g)$
The functions $B'_{k,i}(x)$, $B''_{k,i}(x)$, $B'''_{k,i}(x)$ are the first, second and third derivatives of $B_{k,i}(x)$ respectively.

It is known that the functions $\{B''_{k,i}(x)\}$, $i = -k, -k+1, \cdots, g$, are linearly independent on $[\tilde{a}, \tilde{b}]$; then the matrix D with the (i,j)-th element

$$\int_{\tilde{a}}^{\tilde{b}} B''_{k,i}(x) B''_{k,i}(x) dx$$

is symmetric positive defined. Therefore the system (5) has a unique solution.

When we consider the functional (3) the matrix D in (5) is replaced by the square matrix $\widetilde{D}$ $(g+k+1) \times (g+k+1)$ where

$$\widetilde{D} = D + \beta \begin{bmatrix} M_1 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & M_2 \end{bmatrix} \tag{6}$$

and M_1 and M_2 are square matrices $(k \times k)$ of the form:

$$M_1 = \begin{bmatrix} (\sum_{j=0}^{3} wa_j B^{(j)}_{k,-k}(a) B^{(j)}_{k,-k}(a)) & \dots & (\sum_{j=0}^{3} wa_j B^{(j)}_{k,-k}(a) B^{(j)}_{k,-1}(a)) \\ \dots & \dots & \dots \\ (\sum_{j=0}^{3} wa_j B^{(j)}_{k,-1}(a) B^{(j)}_{k,-k}(a)) & \dots & (\sum_{j=0}^{3} wa_j B^{(j)}_{k,-1}(a) B^{(j)}_{k,-1}(a)) \end{bmatrix}$$

$$M_2 = \begin{bmatrix} (\sum_{j=0}^{3} wb_j B^{(j)}_{k,g-k+1}(a) B^{(j)}_{k,g-k+1}(a)) & \dots & (\sum_{j=0}^{3} wb_j B^{(j)}_{k,g-k+1}(a) B^{(j)}_{k,g}(a)) \\ \dots & \dots & \dots \\ (\sum_{j=0}^{3} wb_j B^{(j)}_{k,g}(a) B^{(j)}_{k,g-k+1}(a)) & \dots & (\sum_{j=0}^{3} wb_j B^{(j)}_{k,g}(a) B^{(j)}_{k,g}(a)) \end{bmatrix}$$

Generally the matrix E, that has a band structure, is ill-conditioned.
The condition number of the coefficient matrix of system (5) depends on the number of knots and their position. However, in this problem it is not essential to develop an automatic routine for finding the minimum number of knots nor their optimal positions. Knots sequences having uniform spacing or coincident with the ordered set of the x_r are used systematically. (The knot placing strategy described in [4] sometimes gives good results.) Generally, the degree k of the spline function, the number of knots, the knots distribution on $[\tilde{a}, \tilde{b}]$ and the weights are given by the designer.

The system (5) can be solved efficiently using the methods described in [2] and [5].
Two iterative methods for determining the regularization parameter γ are presented in [3] and [4]. In order to apply the method [4] the matrix E^TWE must be positive definite; this will be the case if and only if an ordered subset of the data points x_r satisfies the Shoenberg-Whitney condition.

4 Numerical Studies

The method described in the previous section has been implemented in the programming language C and has been included in a "special" interactive system with graphical capabilities for "computer aided cam design".

The advantages accrued by working on an interactive system with graphical capabilities are the very fast turnaround, the immediate graphical display, the simplicity of the control of the parameters that characterize the method (as, the degree of the spline function, the number of the knots and their position, the regularization parameter, the zero end-point derivative constraints,...).

The interactive system has been implemented on a workstation computer SPARC SUN, but it is highly portable and it can run on almost currently available personal computers.

The descriptive aspects of the man-machine dialogue are reduced at the maximum, by designing the interaction to require only very simple actions by the users, which are anticipated by the system. The system has been designed and written with portability in mind. For this the system has been written in C (a FORTRAN version is also available).

In the following example it is shown the effect of varying the position of the knots on a problem that has a total of twenty Kinematic conditions to satisfy. Although cam syntesis problems in which the cam is contrained by a very large number of constraints are rare, the capacity of the system to accomodate these conditions is still important. Often, in cam synthesis problems, certain constraints are introduced by designer, in addition to those that are prescribed by application, to obtain desirable qualitative characteristics in the follower motion and that can easily yield a large array of constraints. The conditions for this problem are listed in the following the table

Cam Angle (degree) x_r	Displacement (cm) $f(x_r)$	Velocity (cm/rad) $f'(x_r)$	Acceleration (cm/rad^2) $f"(x_r)$
0.0	0.00	0.00	0.00
45.0	1.02	-	0.00
90.0	2.00	0.00	0.00
135.0	1.81	-	-1.17
150.0	1.61	-	-
180.0	1.00	-1.20	0.00
210.0	0.39	-	-
225.0	0.20	-	1.20
270.0	0.00	0.00	0.00

We have considered the regularization functional (3) with $wa_j = wb_j = 1$ for $j = 0, 1, 2, 3$ and $\beta = 10^5$. The regularization parameter $\gamma = 1$. The degree of the spline function $\varphi(x)$ is $k = 5$. It is seen that if the number of knots is small, the corresponding fit is not accurate; if it is too large the results are overlay affected by errors in data value. In fact: with 8 knots we obtain a curve that presents large oscillation for the acceleration and jerk, and the velocity constraint in 90 degree is not satisfied. With 10 and 11 knots there are only little differences for displcement and velocity, but with 11 knots there is more oscillations in the acceleration curve and the constrain in 180 degree is not respected.

The results all seem to indicate that $g = 10$ is the optimal number of knots for this example.
The example illustrates how, by varying the knots number , local refinement in the resulting motion can be obtained.

Figures 1-4 illustrate the effect of the knots' placement on the follower displacement, the follower velocity, the follower acceleration and the follower jerk respectively, when we use a uniform spacing of 8, 10, and 11 knots.

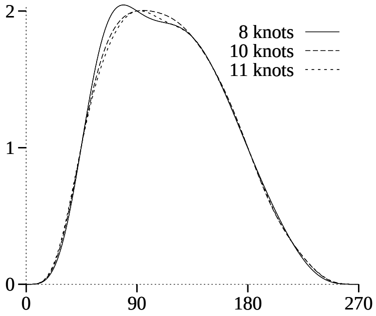

Figure 1: Displacement.

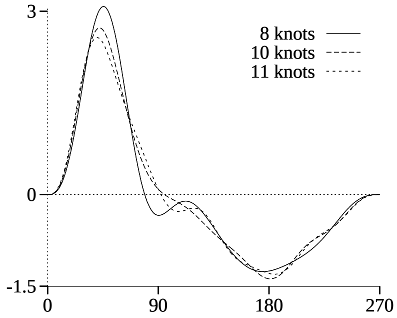

Figure 2: Velocity.

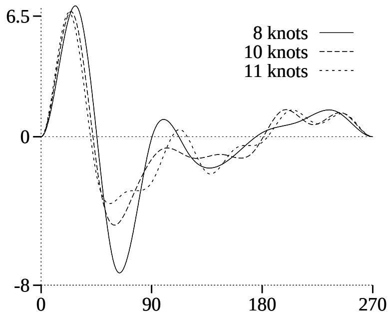

Figure 3: Acceleration.

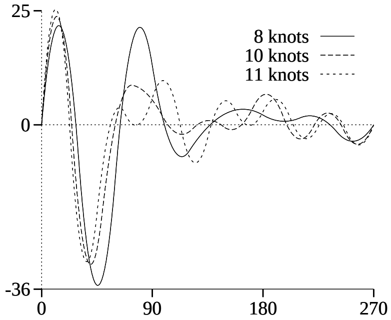

Figure 4: Jerk.

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