An Undergraduate Research Project in Fractal and Chaos using Mathematica

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Introduction. One of today's most exciting areas of mathematics is the study of dynamical systems. There are numerous unsolved problems and the field is extremely active. Not only mathematicians, but also ecologists, chemists, economists, and physicists have become involved in the field. The theory of dynamical systems is used in computer graphics, population models, and meteorology, to name a few. Many mathematicians feel that some knowledge of the subject is imperative. A leading biologist Robert M. May wrote as early as 1976: *I would therefore urge that people be introduced to, say, the Verhulst equation, early in their mathematical education. This equation can be studied phenomenologically by iterating it on a calculator, or even by hand. Its study does not involve as much conceptual sophistication as does elementary calculus. Such study would greatly enrich the student's intuition about non-linear systems. Not only in research, but also in everyday world of politics and economics, we would all be better off if more people realised that simple non-linear systems do not necessarily possess simple dynamical properties.*

At Elizabeth City State University, we are trying to accomplish this through a project "Nurturing Undergraduate Student Researchers (NERT) in Fractal and Chaos" funded by Office of Naval Research. The program focusses on undergraduate education and research training. Nurturing these young researchers is our primary concern. Highest priority is given to providing them with the guidance and skills to ensure their entrance to and success in graduate school. Further, each student in this program learns the fundamentals of scientific research in a structure team setting under the guidance of a faculty mentor. The Fractal and Chaos research team is one of the six teams operating under the NERT grant. In this paper we will explain how this group performed experimental mathematical investigation to acquire some knowledge in the field of dynamical systems. Our approach is to experiment examples, develop intuition from examples, form conjecture, test conjecture, and prove or disprove conjecture.

Definition:

A **discrete dynamical system** is a rule

$$p_{n+1} = f(p_n)$$

that can be used to generate each term of a sequence from the preceding term.

Definition: An **equilibrium point or fixed point** for a discrete dynamical system is a solution of the equation

$$p = f(p)$$

,

that is, a point at which the two curve $y = f(p)$ and $y = p$ intersect each other.

Definition:

We say that a sequence $p_1, p_2, p_3, ...$ **approaches the limit L or converges to the limit** L and write

$$Lim_{n \to \infty} p_n = L$$

if for every very large n, p_n is very close to L.

Given F: $\mathbf{R} \longmapsto \mathbf{R}$ and $x_0 \in \mathbf{R}$ the sequence

$$x_0, F(x_0), F(F(x_0)) = F^2(x_0), F(F(F(x_0))) = F^3(x_0), ..., F^n(x_0), ...$$

is called the orbit of x_0. The basic question one asks in studying iteration is "What happens to orbits over time (as $n \mapsto \infty$)?

First, we look at the following two dynamical systems:

$$P_{n+1} = 2.5P_n(1 - .001P_n), P_1 = 50$$

$$P_{n+1} = 3.5P_n(1 - .001P_n), P_1 = 50$$

Each of these dynamical systems has two equilibrium points, one of which is zero. The non-zero equilibrium point for the first dynamical system is 600 and for the second one is $\frac{2.5}{.0035} \approx 714.29$. Using Mathematica Iterator program in Appendix we find that the limit point of the first dynamical system is 600. In this case, we say that the equilibrium point 600 is "attracting" the limit of the dynamical system. For the second case, however, we find that there is no limit.

Now our goal is to understand completely **when does an equilibrium point attract the limit of a dynamical system?**

First we want to answer this question for simplest of all dynamical systems, the linear dynamical system of the form

$$P_{n+1} = mP_n + b$$

.

We want to observe when does equilibrium point attract the limit point of these dynamical systems?

Experiment # 1: Find equilibrium point for each of the following linear dynamical systems. Use the Mathematica Iterator program to see whether or not the equilibrium point attracts?

1) $P_{n+1} = 0.8P_n + 100, P_1 = 50$

2) $P_{n+1} = 100 - 0.8P_n, P_1 = 50$

3) $P_{n+1} = 0.9P_n + 100, P_1 = 50$

4) $P_{n+1} = -0.9P_n + 100, P_1 = 50$

5) $P_{n+1} = 0.98P_n + 100, P_1 = 50$

6) $P_{n+1} = 1.02P_n + 100, P_1 = 50$

7) $P_{n+1} = 1.2P_n + 100, P_1 = 50$

8) $P_{n+1} = -1.2P_n + 100, P_1 = 50$

9) $P_{n+1} = 100 - P_n, P_1 = 50$

Observation: Based on your experiment with these examples, what appears to determine whether or not equilibrium point will be attracting for the dynamical systems

$$P_{n+1} = mP_n + b?$$

Students were asked to write their conjecture based on their observation.

The following was the student's conjecture:

Conjecture:

For a linear dynamical system

$$P_{n+1} = mP_n + b,$$

the equilibrium point is attracting (or the fixed point is attracting) if $|m| < 1$ and not attracting (or the fixed point is repelling) if $|m| > 1$.

Graphical Analysis

In this section we want to use graphical methods to give us some insight into the question when equilibrium point attracts? Mathematically, we have a function $y = f(p)$. We start with an initial population P_1 and compute the population for subsequent generations by repeating the same procedure; $P_2 = f(P_1)$, then $P_3 = f(P_2)$, then $P_4 = f(P_3)$, etc. We can picture this sequence of events as follows.

STEP 1:

To compute $P_2 = f(P_1)$, start at the point P_1 on the p-axis. Then draw a vertical line up to the graph of $y = f(p)$. Now we are at the point (P_1, P_2).

STEP 2:

To compute $P_3 = f(P_2)$, draw a line horizontally from the point (P_1, P_2) on the curve $y = f(p)$ to the point (P_1, P_2) on the diagonal $y = p$. Then draw a line vertically from this point to the point (P_2, P_3) on the curve $y = f(p)$. This works because $P_3 = f(P_2)$.

Now we can continue this procedure going horizontally to the diagonal line $y = f(p)$, then vertically to the graph of $y = f(p)$ as often as we would like to get a picture of how the population continues to change. We can see using this method whether an equilibrium of a dynamical system attract or not?

We have written a program in Mathematica **Graphical Analysis Program**as given in Appendix.

Using this program students tested their conjecture graphically by doing additional experiments.

Finally, they proved their conjecture which was reformulated as following theorem:

Theorem If $P_{n+1} = mP_n + b$ is a linear dynamical system, $|m| < 1$, and P_1 is any initial value then

$$Lim_{n\to\infty} P_n = \frac{b}{1-m}$$

Notice that $\frac{b}{1-m}$ is simply the equilibrium point for this dynamical system.

If $|m| > 1$ then there is no limit. (Exception: if initial value P_1 is actually equal to the equilibrium point, $\frac{b}{1-m}$, then every P_n is $\frac{b}{1-m}$.

Non-linear dynamical systems are more complicated than linear dynamical systems. Nonlinear dynamical systems may have many equilibrium points, and longterm behaviour of a system may depend on the initial value P_1.

We know that under high magnification near a point, a smooth curve looks very much like the tangent to the curve at the point. The slope of this tangent line is given by the derivative.

Based on the linear case, we have the following theorem for non-linear case:

Theorem: If P_* is an equilibrium point of the dynamical system $P_{n+1} = f(P_n)$, $|f'(P_*)| < 1$, and if P_1 is sufficiently close to P_*, then

$$Lim_{n\to\infty} P_n = P_*$$

;

that is, fixed point P_* is attracting.

Similarly if $|f'(P_*)| > 1$ and if P_1 is sufficiently close to P_*, then the fixed point is repelling.

Malthus population model

This model of population growth is based on the assumption

• The rate of growth of the population is proportional to the size of the population.

Hence the assuption is expressed as differential equation

$$\frac{dP}{dt} = KP,$$

where t = time (independent variable), P = population (dependent variable), K = proportionality constant between the rate of growth of the population and the size of the population.

Students were given U.S. Census figure Funk and Wagnalls 1994 world Almanac from 1790 to 1990 and were asked asked how the solution of Malthus model fits with the actual U.S. population. Using Mathematica, they found out the model of $P(t)$ does a decent job of predicting the population until roughly 1860, but after 1860 the prediction is much too large. Hence the model is fairly good provided the population is relatively small. However, as time goes on, the model predicts that the population will continue to grow without any limits, and obviously, this can not happen in the real world. Next we consider a model which will account for the fact that population exists in a finite amount of space and with limited resources and limited environment.

Verhulst population model We add the assumptions:

• If the population is small, the rate of growth of the population is proportional to its size.

• If the population is too large to be supported by its environment and resources, the population will decrease. That is, the rate of growth is negative.

Hence the assumption is expressed as diffrential equation:

$$\frac{dP}{dt} = K(M-P)P,$$

where t = time, P = population, K = growth rate and M = Maximum supportable population.

Using scaling $p = \frac{P}{M}$ we can change the differential equation $\frac{dp}{dt} = kp(1-p)....(*)$ where $k = KM$.

Discrete version If times are in discrete steps of length Δt, then the corresponding Verhulst model is

$$\frac{\Delta p}{\Delta t} = kp(1-p)$$

For convenience, we choose Δt to be the unit of time, so $\Delta t = 1$. Then we write

$$\Delta p = p_{n+1} - p_n$$

and rewrite equation (*) as $p_{n+1} = p_n + kp_n(1-p_n)$ for n =0,1,2....

Experiment # 2: Comparison between discrete and continuous model

For the choices of $k = 0.5, 1.5, 2.2, 2.5, 2.9$ with $p_0 = .02$ compare the graphical solution of discrete and continuous model using Mathematica.

For k= 0.5, 1.5, students observe that even if discrete and continuous solution converge to the same number, the two solutions may not look alike.

For k = 2.2, they notice something peculiar. The discrete sequence does not get close to any number; instead, within only few iterations, it starts to oscillates back and forth. This limiting behavior is called a cycle of period 2, or simply a 2-cycle. As k increases to 2.5 the iterations settle into an even more complicated pattern, a cycle of a period 4. For k = 3.9, the sequence exhibits no discernible pattern. The values of p seem to jump around at random.

Experiment # 3: Using Mathematica iterator find out what values of k and the initial values of p_0, $0 < p_0 < 1$, the orbit of the discrete dynamical system

$$p_{n+1} = p_n + kp_n(1 - p_n)$$

is

a) simple (converge to 1)

b) interesting (neither simple nor dangerous)

c) dangerous (when the values get larger and larger beyond the computer can handle)

They find that orbits are simple when $0 \leq k \leq 2$, interseting when $2, k \leq 2.57$ (this includes 2-cycles, 4-cycles, 8-cycycles, 16 cycles and so on), chaotic when $2.57 < k \leq 3$ and dangerous when $k > 3$.

Using Pascal program in the Appendix (Mathematica program is time consuming) they capture the bifurcation diagram in the computer.

In studying the dynamics of any function it is important to know its periodic points. At this point students learn a special case of a very beautiful theorem of Sarkovskii, dealing with a period- 3 point.

NEWTON'S METHOD

In this section we turn our attention to study the dynamics of Newton's method. Newton's method is among the most prominent numerical methods for finding solutions of nonlinear equations. Basically Newton's method is an iteration method for computing zero.

Given a differential function f: $\mathbf{R} \longmapsto \mathbf{R}$, Newton's method consists of iterating the function

$$N_f = x - \frac{f(x)}{f'(x)}$$

.

Evidently the roots of f are fixed points of N_f, and we would like to determine the possible behaviors of orbits when N_f is iterated.

Experiment #4 : Using the Mathematica iterator program, determine the set of initial values (say, E) for which Newton's method fails i.e., the points x that $N_f^n(x)$ does not converge to a root of f as $n \mapsto \infty$ for the following polynomials:

a) $f(x) = ax + b$ (linear case)

b) $f(x) = x^2 - 1$(quadratic case with two real roots)

c) $f(x) = x^3 - x$(cubic case with three real roots)

Use more examples for case b) to predict what will happen in general. What happens if the roots are complex for quadratic and cubic case.

The following is the summarized version of students findings for the above experiment:

For linear case all initial values will attract to the fixed point. Hence the set $E = \emptyset$.

After experimenting number of polynomials of type b) we conclude that $E = \{\alpha\}$ where α is the critical point(i.e., $f'(\alpha) = 0$) of $f(x)$. If r_1 and r_2 are the real roots then any initial values in $(-\infty, \alpha)$ will attract to r_1 and any initial values in (α, ∞) will attract to r_2.

Using the Q-basic program in the Appendix we studied the dynamics of the polynomial like $f(z) = z^2 - (3 + 4i)$ by Newton's method . Based on the pictures produced by the Q-basic program , we conjecture that *if the complex quadratic polynomial $p(x)$ has two distinct roots, then the set E is the set of points on the perpendicular bisector of the line segment joining the two roots. Further more, this perpendicular bisector will split the plane into two halves. Each half contains one of the roots. All initial values lie on one half will attract to the root containing that half.* Students were able to conjecture what is known as **Cayley's theorem.**

For the cubic case c) , we found that $\pm\frac{1}{\sqrt{3}}$ are two critical points. By Newton's method, any initial values in $(-\infty, -\frac{1}{\sqrt{3}})$ will converge to -1 , any initial values in $(\frac{1}{\sqrt{3}}, \infty)$ will converge to 1 and any initial values in (a,b) containing 0 will converge to 0. Further analysis shows that the set E contains a sequence $\{a_k\}$ and a two cycle $\{a, b\}$.

Finally for the complex cubic case we developed a program in Q-basic (see Appendix) that would show pictorically using which initial values would attract to which roots, and which would not work. This program check each pixel on the screen and color it based on their outcomes.

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