Asian Conference on Technology in Mathematics
Penang, July, 1997

Computer Visualization and Vector Calculus

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Abstract. This article demonstrates how newly available technology (e.g. the symbolic and graphical capabilities of MAPLE) leads to a complete rethinking how key concepts of (vector-) calculus are introduced. The new approach is accessible to a much broader population of students.
The teaching of single variable calculus has seen compelling changes in the last decade due to new technology. One should proceed in complete analogy on all levels of calculus – here the focus is on vector calculus. Specifically, it is shown how to *"see"* local linearity (differentiability) and (uniform) local constancy (Riemann integrability) through appropriate zooming at all levels of calculus. Highlights are: "How to see the divergence and curl by zooming", and a single, completely "intuitive" argument that applies to Stokes' theorem and all its cousins at all levels.
The proposed approach is implemented in *MAPLE* and has been extensively class-tested.

1 Introduction

The wide availability of inexpensive, graphing calculators has completely changed the introductory presentation of single-variable calculus. In particular, the ability to easily *zoom* in has had a dramatically facilitated the teaching of *local linearity* as the fundamental concept underlying differential calculus.
We propose a radically new way of introducing vector calculus, which takes advantage of computer tools that became widely available only in the last decade. This approach is completely consistent with the modern presentation of single variable calculus. As a special highlight we demonstrate how one can *see* the curl and the divergence provided one zooms correctly, i.e. provided one takes advantage of computer graphics, and consistently implements the idea of local linearity. The highly visual approach is not to replace the analytic treatment, but instead both motivate and guide it.
In addition to unifying all of differential calculus, this approach also much facilitates the presentation of Stokes' theorem in its many versions.

This article does not claim to provide any original mathematics. Instead it demonstrates how newly available technology leads to a complete rethinking of the presentation of very classical material. The first two sections review the impact of zooming for the introduction of single-variable differential calculus, and the often unclear use of zooming

[1]This work was partially supported by the NSF-sponsored Foundation Coalition through Cooperative agreement NSF EEC 92-21460. and by the ACEPT through NSF DUE 94-53610

of rcontinuity and integration. We identify two completely different ways of zooming! We proceed through functions of several variables to zooming on vector fields, e.g. asociated with line integrals. The bulk of the new material is contained in the last three sections that deal with derivatives of vector fields, linear vector fields, and with Stokes' theorem.

We have impkmented these kinds of zooming into easy-to-use MAPLE procedures, see [3]. However, they may be easily adapted to e.g. *MATLAB* or *MATHEMATICA* or similar packages. We are in the process of class-testing for a second time a large library of interactive modules of associated in-class exercises. The MAPLE worksheets in their preliminary form are all accessible on the World Wide Web [4].

Since this approach to vector calculus, while so obvious and consistent, nonetheless is radically different from traditional presentation, we may expect that it will take some time to fine-tune the exercises and projects that are the heart of modern classes, as well as finding the links to the best matching applications for each step of this developments. In this paper we only give a few ideas, and survey some assignments that we currently use in our classes. A comprehensive *picture-book* in about 80 articles on *Limits and zooming, from continuity to Stokes' theorem* [3] will soon appear elsewhere. Continuous refinements of our in-class exercises, as well as links to related work by other authors, will be made available on the WWW at above address.

The book [3] also conatins the programs for a variety of animations resulting from integration of curl and divergences, as well as for appropriately colored animations of Frenet frames etc for the calculus of space curves. In this article, however, we shall focus on the different kinds of zooming, and in particular, how one can see the curl by zooming.

2 Zooming and slope

Until the mid-eighties almost all students learnt that the slope of a *tangent line* is the graphical analogue of the limit of the difference quotient

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

This line is argued to be, in some *vague* sense, the limiting position of the secant lines that pass through points $(a, f(a)$ and $(a+h, f(a+h))$ on the graph of the function. Typically a single, static illustration, like figure 2.1, was provided by text-books.

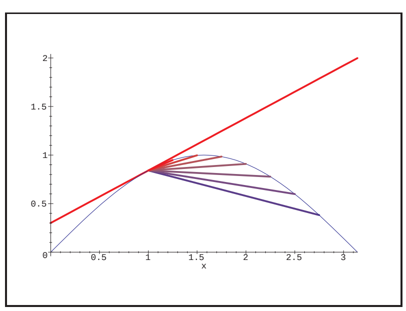

Figure 2.1 Traditional secant lines

This picture has many drawbacks: It is a static image that is supposed to illustrate the *process* of taking a limit. Many students concentrate on the line segment from $(a, f(a))$ to

$(a+h, f(a+h))$, rather than on the secant line, and this segment disappears in the limit! The educational community also has found substantial confusion among students about the concept of a tangent line. For example, students often have the misconception (their *"definition"*) that tangent lines can only touch the graph in one point (leading to mistakes with functions like $f : x \mapsto x^2 \sin(1/x)$ at $x = 0$) Finally, with such static pictures the applicability of *local linearization* techniques often remains unconvincing as pictorially the quality of the approximations appears to be very poor.

The advent of inexpensive graphing calculators has completely changed the scene. Practically every calculus student in the past decade has gone through a zooming-exercise in some form or another. In my classes, I use personal computers and the program *FORMULA TUNE* of the free package *ARIZONA SOFTWARE* [6]. Using cursors students may reposition the point of zooming, and single key-board strokes `z` or `Z` zoom in and out, respectively. The class picks a generic (algebraic) formula. Every student is assigned a different point. The students zoom in repeatedly, leave their final image on the screen, and wander through the class, to discover that everyone ends up with a straight line that is characterized by a single number, its slope, compare figure 2.1. Pixel for pixel the "limit" is reached within a finite number of steps (which only depends on the resolution of the screen).

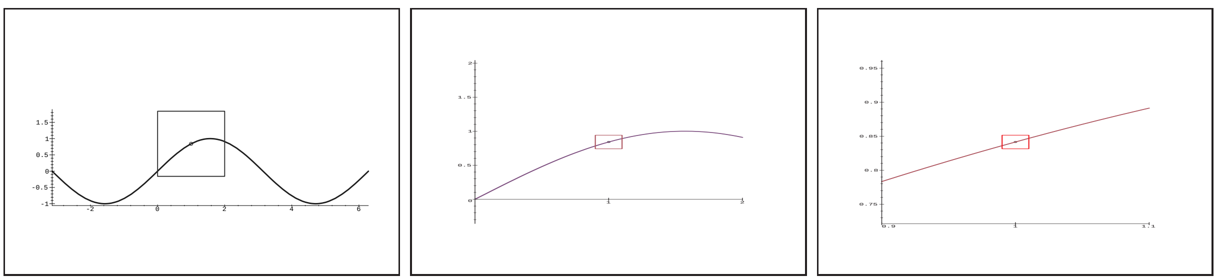

Figure 2.2 Zooming in single-variable calculus

The advantages of this visualization are manifold: This process goes right to the heart of the most fundamental idea on which all differential calculus rests – local approximability by a linear function. This last picture is much closer to the characterization of differentiability that avoids the difference quotient, and emphasizes local linear approximability.

Definition. A function $f : X \to W$ between normed linear spaces X and Y is differentiable at $p \in X$ if there exists a linear map L_p such that

$$\|f(p+x) - f(p) + L_p(x)\| = o(\|x\|) \tag{2}$$

The process of taking a limit is now visualized again by a *process*, by repeated *zooming*. The images are very compelling,. All available evidence suggests that even people who no longer actively work with mathematics remember for life this fundamental concept of local linearity that underlies all of differential calculus.

We conclude this review with a few remarks about the fundamental theorem. Students typically spent considerable amounts on time in secondary schools working with linear functions, with slope, ratios of *"rise over run"*, linear extrapolation etc. The telescoping sum

$$f(b) - f(a) = \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1}) \right) = \sum_{k=1}^{n} \left(\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}} \right) (x_k - x_{k-1}) \qquad \text{for } a = x_0 < x_1 < x_n = b \tag{3}$$

intuitively is the basis for developing the fundamental theorem of single variable calculus. One argues that in the nonlinear case (for sufficiently fine partitions $a = x_0 < x_1 < x_n = b$)

$$\begin{aligned} f(b) - f(a) &= \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1}) \right) \\ &= \sum_{k=1}^{n} \left(\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}} \right) (x_k - x_{k-1}) \\ &\approx \sum_{k=1}^{n} f'(x_k) \, (x_k - x_{k-1}) \\ &\approx \int_a^b f'(t) \, dt \end{aligned} \tag{4}$$

Most traditional textbooks employ the mean value theorem to make this into a formal proof under suitable hypotheses. We prefer to go a different route that relies on notions of uniform continuity and uniform differentiability instead of utilizing the mean value theorem, compare [9]. The key advantage is that these proofs generalize in a straightforward fashion to all versions of Stokes' theorem, in all dimensions and on manifolds. We argue below that with modern computer visualization the uniform notions are much more accessible, at lower levels than before, and even to students with much less formal mathematical training.

The most natural requirements on the function f above, is that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition $a = x_0 < x_1 < x_n = b$ with $\max_k \{|x_k - x_{k-1}|\} < \delta$ the error in each of the approximate equations of (4) is bounded by ε. Thus a natural requirement is that f is uniformly differentiable, and that the derivative is uniformly continuous on the interval $[a, b]$. In the discussion of a completely revised vector calculus, we will utilize the corresponding hypotheses. These naturally lead to an *obvious* generalization of the proof of the fundamental theorem to the standard versions of Stokes theorem.

3 Continuity, and zooming

Modern technology, in the form of simple zooming, has had a dramatic impact on how differentiation is first introduced to today's calculus students. Surprisingly, the situation regarding continuity and integration is much muddier. There are again two pictures for visualizing the ϵ-δ criterion for continuity. One of them, which we refer to as *"shrinking"* is analogous to the classical secant lines approaching the tangent lines while the distance between the two points shrinks to zero – compare figures 3.1.a and 3.1.b.

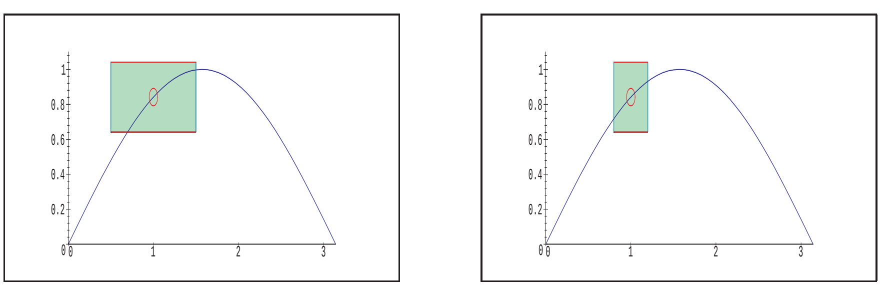

Figure 3.1 Continuity and shrinking boxes

Loosely speaking, the function f is continuous at a, if for every vertical window size 2ε one can find a horizontal window size 2δ such that the graph of f exits the window (centered at $(a, f(a))$) through the sides (without touching the bottom or top). (More precisely, the graph of the restriction of f to the part of the domain corresponding to the horizontal window size must lie entirely within the window.)

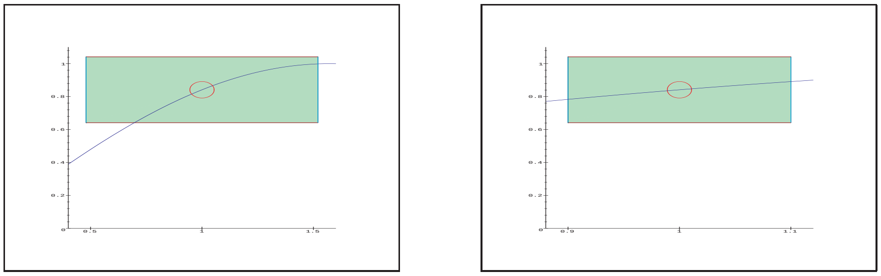

Figure 3.2 Continuity by zooming of zeroth kind.

Again, the alternative is to magnify, to zoom in, compare figure 3.2. Thus we distinguish at least two different forms of zooming: For differentiation one needs to magnify domain and range at the same rates. For continuity one needs to fix a vertical window size (characterized by ε) and only magnify the domain (the horizontal window-size characterized by δ). We refer to these two kinds of zooming as **"zooming of the first kind"** (to *"see"* **"local linearity"**), and as **"zooming of the zeroth kind"** (to *"see"* **"local constancy"**). (Further notions of zooming, e.g. using quadratic rates at critical points are utilized extensively in [3]).

While there are places where it is appropriate to think of shrinking, we plead for more consistency, and a systematic use of zooming throughout all levels of calculus! For example, the standard images for Riemann sums as collections of rectangles that *"exhaust"* an area are more compelling if the widths of the rectangles shrink to zero. However, the argument that the integral (the limit of the Riemann sums) exists, relies on the notion of uniform continuity, i.e. uniform local constancy! Briefly, for any given $\varepsilon > 0$, by uniform continuity there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(2(b-a))$ whenever $|x - y| < \delta$ (and $x, y \in [a, b]$). Pictorially, choose the horizontal window size to be δ, showing some subintervals $[x_{k-1}, x_k]$. No matter what subdivision of the shown subinterval $[\alpha, \beta]$ (and which points $\xi_k \in [\alpha, \beta]$) one chooses, the corresponding Riemann sum obviously lies within *veps* of the original sum. From here it is a straightforward argument in terms of Cauchy sequences. While the formal proof is typically not encountered until a junior level course in advanced calculus (in the US), the pictorial arguments utilizing zooming are very compelling, accessible to a much broader population, and mathematically they are completely sound! For further details see again [3].

We conclude this section with a few notes on uniform continuity, compare also [9]. Standard integral calculus at the introductory level is (for good reason) are only concerend with integrals over compact sets (closed bounded intervals or subsets of the plane or $\mathbf{R}^3$). (Improper integrals over the entire space really utilize σ-compactness.) The standard assumption is piecewise continuity of the integrand. On compact sets continuity implies uniform continuity, which, as argued in the previous section is the basis for the most natural proofs of the existence of Riemann integrals, and of proofs of the fundamental theorems.

On a purely symbolic level working with three quantifiers is considered quite challenging.

$$\text{(C)}\quad (\forall\varepsilon > 0)\,(\forall x \in [a,b])\,(\exists\delta > 0)(y \in [a,b], \|y - x\| < \delta \Rightarrow \|f(y) - f(x)\| < \varepsilon)$$

$$\text{(UC)}\quad (\forall\varepsilon > 0)\,(\exists\delta > 0)\,(\forall x \in [a,b])(y \in [a,b], \|y - x\| < \delta \Rightarrow \|f(y) - f(x)\| < \varepsilon)$$

It is no surprise that traditional calculus books (that are almost entirely symbol-based) refrain from discussing (uniform) continuity in great detail. On the other hand, pictorially, it is very easy to teach both notions to students even at very early levels. Indeed, here the *tracing* capability built into most graphing calculators is most valuable: After the ε-δ discussion suggested above, the question of uniformity is often raised by the students themselves!

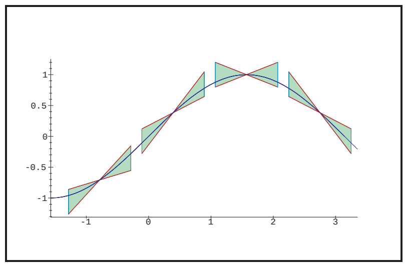

Figure 3.3 Uniform differentiability

After finding suitable pairs of matching vertical and horizontal window sizes, trying to trace the graph of the function it is most natural to ask whether, for any predetermined vertical window size 2ε, one may find a horizontal window size 2δ that, *"works"* at all points. (It is very awkward to repeatedly change the width of the window as one traces the curve.) Even in introductory classes it is not too farfetched to expect that with well-designed exercises students will make a conjecture like: "On closed bounded intervals continuity implies uniform continuity", as well as give counterexamples with explanation of what may go wrong on either unbounded intervals, or on intervals that are not closed.

We claim that the uniform notions are not only much more natural, and more appropriate for introductory courses (where we follow many great mathematicians, e.g. more recently Lax [5] and Stroyan [9]), but with modern computers/graphing calculators, the uniform notions are also much easier to teach and understand!

4 Functions of two variables

In complete analogy to the single-variable case, before getting to calculus students extensively review linear functions of two variables. They review the notions of two indepent slopes of a plane, the normal vector to a plane, and the volume of a right prism (a region in $\mathbf{R}^3$ of the form $P = \{(x,y,z) : (x,y) \in R, a \leq z \leq b\}$, where R is a nice region in the plane, usually with piecewise smooth boundary). Properly interpreted these are derivatives of linear functions, and integrals of constant functions.

In traditional textbooks it is not always made very clear how these basic ideas generalize to to derivatives of nonlinear functions and to integrals of nonconstant functions. More

recent texts like [2] emphasize links between symbolic, (numeric,) and graphic avenues. [2] is particular, make extensive use not only of graps of functions of two variables, but also of contour diagrams.
We ascertain that zooming of any kind (zeroth, first, and second order) works just as well for graphs and contour diagrams of functions of two variables as it does for the single variable case. We leave the details as an exercise for the reader (who may also refer to [3] for a complete description). We encourage the reader to carefully distinguish between more common classical pictures corresponding to shrinking (e.g. tangent planes), and those which correspond to zooming. E.g., what does a contour diagram look like after zooming of the first kind, of the zeroth kind? More specifically, the slopes become partial derivatives, while the normal vector will lead to the notion of the gradient. The convergence of Riemann sums again is an immediate consequence of uniform continuity, i.e. uniform *"local constancy"*.

Moreover, the uniform notions make just as much sense in this case as well. We note that the most interesting generalization of the fundamental theorem for multi-variable functions leads to line integrals of gradient fields, and we leave the detailed discussion of this to the next section.

5 Vector fields and zooming

Following traditional abuse, we will not distinguish between vector fields and covector fields, or between one-forms and two-forms in 3-space. Our goal is to utilize modern technology in innovative ways to foster a better understanding of the key concepts of calculus, and the language that has been established in the last century, and that students need to master in order to communicate with the rest of society

What does one obtain if one naively zooms in on a vector field (represented by arrows) using any of the standard software packages (computer algebra systems or various educational packages)?

The image of a constant vector field, i.e. parallel arrows of equal lengths may appear boring, but it is at the heart of numerical algorithms for integrating differential equations, starting from Euler's method. For a detailed discussion of this point of view, the reader is referred to [3].

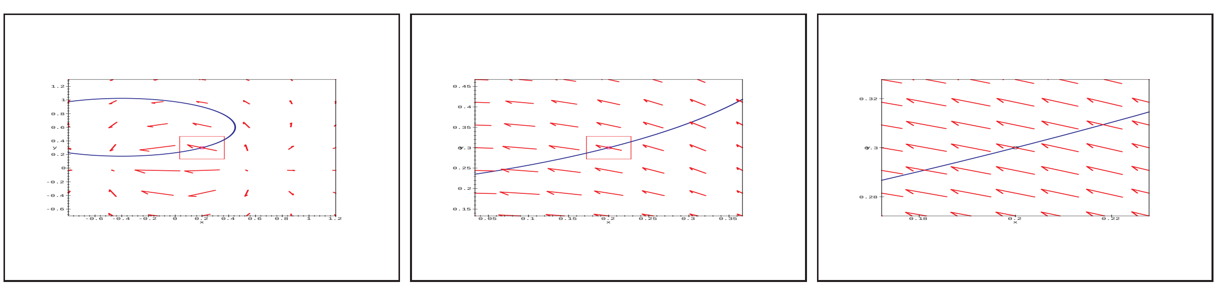

Figure 5.1 Zooming for line integrals

Very similar are the pictures corresponding to line-integrals: Consider a given (image of a) smooth curve, superimposed vector field (easier to visualize in the plane). Fix a point on the curve and *naively* zoom in. The result is a straight line segment in a constant vector field. Again, this may not appear exciting. Yet again, this is at the very heart of the concept of line-integrals. Using the language of mechanics, the work done by moving along a straight line segment $\vec{\Delta s}$ in a constant force field $\vec{F}$ equals the dot-product $\Delta W = \vec{F} \cdot \vec{\Delta s}$.

The work done by moving along a (piecewise) smooth curve in a general (continuous) nonconstant field is approximated by a Riemann sum $W = \sum_k \Delta W_k = \vec{F}(x_k) \cdot \vec{\Delta s}_k$. Most naturally again using hypotheses of (piecewise) uniform smoothness of the curve, and of uniform continuity of the vector field, one easily shows that these Riemann converge as the partitions become increasingly finer.

After short reflection it becomes clear that it makes a lot of sense what one receives after above naive zooming: The vector field is to be integrated, and the natural hypothesis is local constancy (continuity). The direction of the curve matters. Indeed, standard evaluation procedures relying on parameterizations require to differentaite the curve. This is in complete agreement with arriving at a straight line segment after zooming! A mixture of zooming of the zeroth kind and zooming of the first kind.

On the side we note that the analogous pictures are easily developed for flux integrals over smooth surfaces in 3-space, for a detailed exposition see [3]. That reference also treats in detail how (uniform) continuity of vector fields may be visualized using an appropriate form of zooming.

This mental images resulting from zooming on vector fields lead in a natural way to the fundamental theorem for line-integrals of conservative vector fields, that is, vector fields $\vec{F}$ that are gradients of a potential function φ. First consider a linear function φ and thus a constant field $\vec{F} = \vec{\nabla}\varphi$. If the curve is a straight line segment, the integral immediately reduces to the single-variable case. Slightly more interesting are polygonal curves, which technically are almost the same, but which lead to less trivial, or more interesting telescoping sums:

$$\begin{aligned} \varphi(b) - \varphi(a) &= \textstyle\sum_{k=1}^{n} \left(\varphi(x_k) - \varphi(x_{k-1})\right) \\ &= \textstyle\sum_{k=1}^{n} \left(\frac{\varphi(x_k) - \varphi(x_{k-1})}{\|(\vec{\Delta s})_k\|}\right) \|(\vec{\Delta s})_k\| \end{aligned} \tag{5}$$

where $a = x_0, x_1, \ldots x_n = b$ are the (ordered) vertices along (the image) of the curve.

For the general case consider a (uniformly) continously differentiable parametrized curve $\gamma : [t_0, t_1] \mapsto \mathbf{R}^n$ and a (uniformly) continuous vector field $\vec{F}$ (uniformly on an open region in $\mathbf{R}^n$ containing the image $C = \gamma([t_0, t_1])$ of the curve). The existence of the line-integral $\lim \sum_k \vec{F}(x_k) \cdot \vec{\Delta s}_k = \oint_C \vec{F} \cdot \vec{ds}$ was discussed above. Again the telescoping sum (5) most suggestively generalizes to the general case.

$$\begin{aligned} \varphi(b) - \varphi(a) &= \textstyle\sum_{k=1}^{n} \left(\varphi(x_k) - \varphi(x_{k-1})\right) \\ &= \textstyle\sum_{k=1}^{n} \left(\frac{\varphi(x_k) - \varphi(x_{k-1})}{\|(\vec{\Delta s})_k\|}\right) \|(\vec{\Delta s})_k\| \\ &\approx \textstyle\sum_{k=1}^{n} \vec{\nabla}\varphi(x_k) \cdot \|(\vec{\Delta s})_k\| \\ &\approx \oint_C \vec{F} \cdot \vec{ds} \end{aligned} \tag{6}$$

Using the uniform continuity and differentiability hypotheses, it is again straightforward to go from approximate equations to equations in the limit as the subdivision becomes arbitrarily fine.

The pictures discussed here can be found in various places – but they are rarely completely exploited. This naive kind of zooming on vector fields (and possibly superimposed curves) corresponds to integration of vector fields. It related to local constancy, continuity, and thus the appropriate name is *zooming of the zeroth kind*. This raises the question of how to zoom of the first kind on vector fields, so as to discover their derivatives? This question brings us to the next section.

6 Derivatives of vector fields by zooming

What is zooming of the first kind mean for vector fields? It is clear that the resulting images should reveal divergence and curl at the least! The logical order to be followed in a class postpones this section until after the coverage of linear vector fields and their characteristics. However, to keep the flow of this article we violate this natural order and jump ahead.

Recall the distinction between zooming of the first kind, and zooming of the zeroth kind: When zooming of the zeroth kind, that is for continuity and integrals, one only magnifies the domain. This is what the last section was about. When zooming of the first kind, that is for derivatives, one needs to magnify both domain and range (at equal rates). The difficulty with this kind of zooming (that apparently has kept people from utilizing it since adequate computer software became available about a decade ago) is the representation of the *graph* of a vector field by arrows.

The key observation (which immediately will suggest the entire approach), is that when zooming of the first kind, one cannot expect to keep the first axis (representing the origin of the range) in view. Thus it is natural to subtract the constant field $\vec{F}(p)$ from the vector field $\vec{F}(x)$ (when zooming on $\vec{F}$ at the point p). Next zoom in on the vector field by further and further restricting the domain (containing the point p in the center). Incidentally, most common software packages will automatically rescale the lengths of the arrows drawn on the screen. Technically this may lead to some initial mistakes at the beginning. However, these effects are minor, and the advantage of getting (almost) the right thing for free are so inviting that most users won't bother.

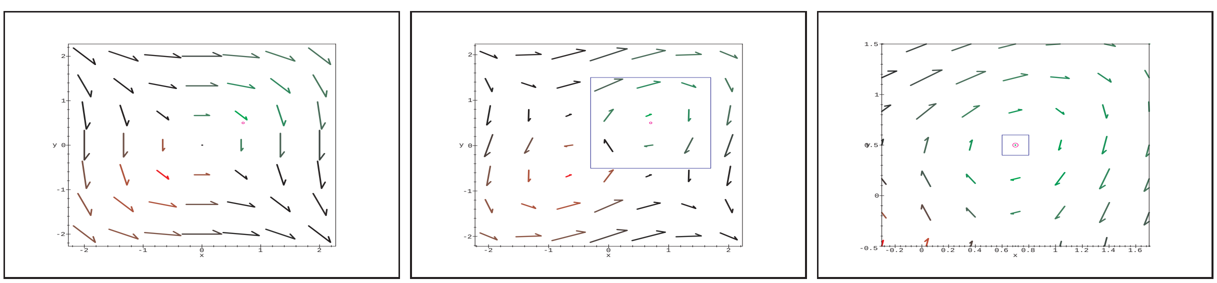

Figure 6.1 Zooming of first kind for vector fields

What is the result of this zooming of the first kind on a vector field? It clearly should yield the local linearization of the vector field (the purely linear term, after subtracting the constant drift term). This linearization clearly contains all derivative information at the point p – it remains to unravel the details. Since this is such a new approach, we devote the entire next section to this task.

Before proceeding, we make a few remarks about the geometry: One of the particularly nice features of this approach is that it leads to a coordinate-free, i.e. geometric, notion of the derivative(s) of a vector field! However, there are some tricky features: In a more general setting, the vector field may be a smooth *tangent* vector field on a smooth manifold. In general there is no notion of *constant vector field* on a manifold, and even less of a *linear vector field*. It requires the additional structure of a connection that only will allow one to generalize the above zooming to this truly nonlinear geometric setting. Intuitively this allows to identify (*transport*) tangent vectors $F(x) \in T_xM$ that are elements of the varying tangent space at $x \in M$ with tangent vectors that are elements of the fixed tangent space T_pM. This setting also sheds some additional light on the true nature of the rescaling of the vector field as one zooms in. The objects in the tangent space are *infinitesimal* objects, and their *lengths* are not comparable to distances on the manifold (assuming a Riemannian structure).

7 Derivatives of linear vector fields

Zooming of the first kind yields a linear vector field. This is the derivative of the original vector field at the point of zooming. However, for practical purposes, and physical applications, one typically is interested in more condensed information – in electromagnetic fields as well as fluid flow the most important quantities are the divergence and the curl.

Modern technology, with this zooming, invites to characterize these two quantities in a way that is extremely close to the derivatives of single- and multi-variable calculus: The slope that characterizes a line is the ratio of rise over run $m = \frac{y_2 - y_1}{x_2 - x_1}$. The key feature of linearity is that this ratio is independent of the choice of the two points (x_1, y_1) and (x_2, y_2) on the line. The normal vector (normal to the contours in the domain) that characterizes a plane is characterized by pointing in the direction of steepest increase, and by its magnitude which is the rate of increase in this direction. Again, linearity means that one obtains the same normal vector at any point of the domain (or of the plane).

We intend to follow the very geometric treatment of divergence and curl that may be found in the classic text on Electricity and Magnetism by Purcell[7]. But we feel that it is just as crucial to completely understand the linear case, before going to the nonlinear case. Recall, that students typically spend years working with linear functions and the slopes of their graphs, before they learn that the derivative of a nonlinear function at a point is (the slope of the graph of) the linear function seen after zooming. Similarly, every calculus book reviews planes, their slopes and normal vectors before proceeding to derivatives of nonlinear functions. Their partial derivatives and gradients are then defined as the slopes and normal of the linear function (plane) seen after zooming at the point of interest. However, in the case of vector fields most textbooks immediately start with the nonlinear case – as if vector fields themselves were not yet hard enough new objects for most students!

In our classes we use applications such as flow of incompressible fluids, and gradient fields to motivate a rigorous formalization of divergence and rotation. In some sense we appeal to the students' intuition to postulate criteria such as zero net flux across a closed curve or surface for incompressible fluid flow, and zero work in conservative force fields along closed contours, as well as zero *elevation change* along closed curves in gradient fields. With modern software these very pictorial arguments are easily accessible to a very large population of students.

Some well-spent time should be devoted to a discussion of line-integrals over constant vector fields, and to an in-depth discussion of linearity of the line-integral. It is essential that everyone understands that the contribution of any constant vector field to any line (or flux) integral over any closed curve (surface) is zero. This is intimately related to the subtraction of the constant part when zooming of the first kind.

The next step is to actually carry out some calculations. Fix a linear vector field in the plane such as $\vec{F}(x,y) = (13x - 9y)\vec{\imath} + (11x - 3y)\vec{\jmath}$. Each student is assigned a polygonal or circular or similar simple contour in the plane. Students set-up their line integrals and evaluate them with pencil and paper, numerically using computers, or using computer algebra systems, and report their findings to a transparency on an overhead in the front of the class. With properly chosen coefficients such as the field above, it takes only very little discussion, and students will observe that the integrals are independent of the location and the shape of the contour, and are only scaled by the area of the region enclosed by the contour. The class repeats the exercise with two more vector fields and more contours leading to similar observations. Students typically conjecture a relation between the ratio line integral divided by area and the coefficients of the linear vector field.

Next the conjecture is to be made into a theorem: The ratio of the line integral $\oint_C \vec{F} \cdot \vec{N}\, ds$ divided by the area of the region enclosed equals the trace of the linear vector field $\vec{F}$. First an anlytic proof is given for rectangles aligned with the coordinate axes. The structure of the proof is very similar to the usual proof of Green's theorem for simple regions in the plane. However, in this linear setting the proof does neither require any limits nor any serious calculus. Indeed, for linear integrands over line segments both the trapezoidal and the midpoint rule are exact! The calculation is sketched for the circulation integral over a rectangle centered at (x_0, y_0) with width $2\Delta x$ and height $2\Delta y$, and the vector field (using the midpoint rule) $\vec{F}(x,y) = (ax + by)\vec{\imath} + (cx + dy)\vec{\jmath}$

$$\begin{array}{rcl} \oint_C \vec{F} \cdot \vec{T}\, ds & = & \vec{F}(x_0, y_0 - \Delta y) \cdot \vec{\imath} + \vec{F}(x_0 + \Delta x, y_0) \cdot \vec{\jmath} \\ & & + \vec{F}(x_0, y_0 + \Delta y) \cdot (-\vec{\imath}) + \vec{F}(x_0 - \Delta x, y_0) \cdot (-\vec{\jmath}) \\ & = & (c - b) 4 \Delta x \Delta y \\ & = & (c - b)(\text{ area of the rectangle }) \end{array} \tag{7}$$

(Some may prefer to work entirely geometrically, and a coordinate-free calculation over arbitrary triangles is not too hard.) From here, it is a few exercises to get the same result first for right triangles, then for arbitrary triangles, and finally for polygonal curves. This calculation is still entirely without limits or serious integrals! The step to polygonal curves is the best place to introduce the new variation of *"telescoping sums"*: Now the region inside the curve is triangulated, and the usual arguments apply to conclude that the net contribution of all line integrals over all interior edges cancel, but we are still in the limit-free case!. The final step from polygonal curves to smooth curves involves limits – but these limits involve mainly the bounding curve and the region inside, not the vector field. This development splits the proof of Stokes' theorem and its cousins into an essentially limit-free development that allows one to concentrate on the new objects and the new arguments (e.g. new *telescoping sums*) – the proof of the nonlinear Stokes' theorem then will exactly correspond to the argument made for the fundamental theorem.

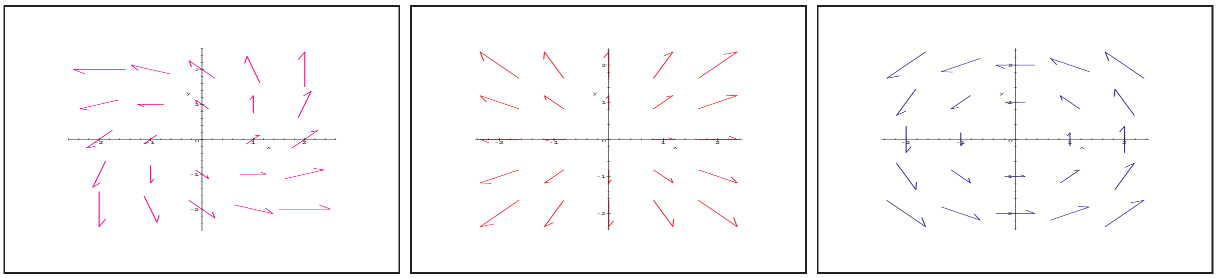

Figure 7.1 Splitting a vector field into symmetric and skew part

During this development of the trace and the skew symmetric part determining the values of circulation and flux integrals, it becomes second nature to always split any linear vector field into its symmetric part and its skew symmetric part. Until recently, without adequate computer technology very few students had the privilege of having graphical tools available for working with vector fields – but now it is very easy, and the illustrations provide very compelling links to objects encountered on a daily basis (e.g. weather forecasts showing wind patterns that are rotating on a continental scale). This will be important when introducing the curl in three dimensions.

8 More derivatives and Stokes' theorem

With this preparation that strongly relies on visualization employing modern computer software, the definitions of the derivatives of vector fields follow exactly the now standard development of the derivative in single variable calculus using zooming.

More specifically, the derivative of a vector field $\vec{F}$ at a point p is the linear vector field $\vec{L}$ seen after (infinite) zooming of the first kind. The divergence of $\vec{F}$ at p is defined as the trace of this linear vector field $\vec{L}$, while the rotation and curl are defined via the skew symmetric part of the field $\vec{L}$. (Usually, in 2 dimensions the rotation is taken to be scalar, while in 3 dimensions the curl is identified as a vector. Geometrically, the curl could as well be taken to be a section of an $so(3)$ bundle, or as a 2-form. What matters here is only the use of modern technology that opens a completely new approach that is both sounder, and more in line with both single-variable calculus, as well as functional calculus.)

As a *" check for understanding"* we revisit the exercise on line integrals discussed in the previous section. Now we take a rather generic formula for a vector field in the plane or 3-space (taking care **not** to accidentally pick a linear, divergence-free or irrotational field). Now we put students together in groups, and assign to each student a different contour, based at the same point. These may be rectangles, triangles, combinations of circular and polygonal arcs, and in various positions relative to the common base point. Again each student sets up and evaluates the line integral over his/her contour, and they

report their findings to a transparency on an overhead projector in the front. This time the results show no dominant pattern – even after rescaling by the area of the enclosed region.
However, this time the students are asked to repeat the exercise several times after shrinking their contour towards the common base-point by factors of 10, 100, 1000, etc. A clear pattern emerges of the repeated results, and apparently this time the ratio of line integral divided by area approaches a limit as the contours shrink to a point. This is contrasted with zooming (of the first kind) on the vector field at the base point (together with the sequence of the superimposed contours of all sizes). As expected, upon sufficient zooming the vector field appears more and more linear and the ratios approach a limit!

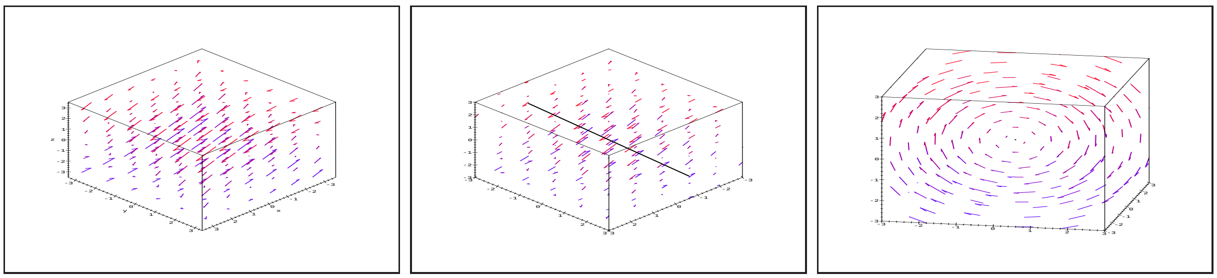

Figure 8.1 Seeing the curl by zooming of first kind

After in-depth work with zooming for derivatives of vector fields in the plane, the step to three dimensions yields a surprise: Usually it is very hard to obtain useful plots of vector fields in three dimensions. As an in-class exercise, again we fix a rather generic formula for a vector field in three dimensions. Every student is assigned a point in three space (e.g. determined by the student's birthday). Each student zooms in (of the first kind on the vector field at hie/her personal point – i.e. first subtracts the constant part. As has become common practice in the plane, after zooming, the students are to view the symmetrie part (nothing exciting) and the skew symmetric part separately. At first glance the latter appears still quite patternless – yet after a little jiggling it becomes apparent that there is order. Very quickly every student uses the controls to rotate the *box* so that it appears that one is looking down a tube. At every point they discover a rigid rotation – its axis and strength are the curl. This experience appears to be even more compelling than the zooming of single variable calculus – and it makes a concept tangible that for many students usually remains hidden behind many partial derivatives.

After this preparation employing visualization and modern computer technology, the proofs of Stokes' theorem and its cousins are just as intuitive as is the proof of the fundamental theorem in single-variable calculus. In the linear case the key step is just a telescoping sum – after multiplying and dividing by the lengths/areas/volumes of the partition elements. The ratios rise over run, or line/flux integral divided by area/volume are part of the definitions of the derivative. In the intuitive first step one simply changes from equations to approximate equalities for sufficiently fine partitions. Alternatively, one may utilize a little-oh notation – but this requires the hypothesis of uniform continuity of the integrand, and of uniform differentiability of the vector field.

For the sake of simplicity we illustrate the argument for the proof of Green's theorem in the plane for uniformly continuously differentiable vector fields, and planar regions bounded by smooth curves. First rewrite the *telescoping sum* of the linear case using approximate equalities:

Suppose R is a simply connected region in the plane that is bounded by a uniformly differentiable curve C. Partition the region R into a collection of regions R_k (that are nonoverlapping except for their common boundaries) each having a piecewise uniformly differentiable curve C_k as boundary. Denote the areas of the regions R and R_k by A and A_k, respectively.

For each region R_k fix a point $p_k \in R_k$ and let $\vec{L}_k$ denote the lineariztion of $\vec{F}$ at the point p_k (e.g. the linear field seen after infinite zooming of the first kind). The earlier *telecoping sum* identities together with the definitions of the rotation/divergence then become

$$\begin{aligned}\oint_C \vec{F}\cdot\vec{N}\,ds = \sum_k \oint_{C_k} \vec{F}\cdot\vec{N}\,ds \quad &\approx \quad \sum_k \oint_{C_k} \vec{L}_k\cdot\vec{N}\,ds \\ &\approx \quad \sum_k \mathrm{tr}\vec{L}_k\, A_k \approx \sum_k \iint_{R_k} \mathrm{div}\vec{F}\,dA = \iint_R \mathrm{div}\vec{F}\,dA\end{aligned} \tag{8}$$

It remains to analyze the error bounds for the two approximate equalities: Let ℓ_k and r_k denote the lengths of the curves C_k and the diameters of the regions R_K, respectively. First use the geometric definition of the divergence and the uniform differentiability of $\vec{F}$. Given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that $\|F(p)-L_k(p)\| < \varepsilon/(20A)\cdot r_k$ whenever $p \in R_k$. We restrict the partitions to those of sufficient regularity, and only consider partitions that are such that $\ell_k r_k < 10A_k$ for all k. Thus $|\oint_{C_k}(\vec{F}-\vec{L}_k)\cdot\vec{N}\,ds| < \varepsilon/(2A)\cdot A_k$ whenever R_k is a region of diameter less than δ_1 that contains p_k.

Using uniform continuity of $\mathrm{div}\vec{F}$, and noting that $\mathrm{tr}\vec{L}_k = \mathrm{div}\vec{F}(p_k)$, there exists a $\delta_2 > 0$ such that $|\mathrm{div}F(p) - \mathrm{div}F(q)| < \varepsilon/(2A)$ whenever $|p-q| < \delta_2$. Thus the combined error in both approximate equalities is bounded by $B = \sum_k 2\,(\varepsilon/(2A)\cdot A_k) = \varepsilon/2$ whenever the partition is chosen such that the maximum diameter r_k does not exceed $\min\{\delta_1, \delta_2\}$, completing the proof.

This approach is mathematically completely sound. The proof most commonly found in textbooks relies on an almost purely algebraic inductive argument that reduces the proof of Stokes' theorem in dimension n to Stokes' theorem in dimension $(n-1)$, and eventually to the the fundamental theorem. Our alternative directly employs the definition of the derivative in any dimension n, which in turn is shown to immediately arise from the concept of local linearity. The key steps in the proof are practically identical in all dimensions, and they are very intuitive! We prefer to base the arguments on the hypotheses of uniform continuity and uniform differentiability, which we claim are most natural in this setting. The only well-known difficulty arises from dealing with pathological surfaces in dimensions three and higher. Without going much into details here, our preference is to explicitly require compactness and uniform differentiability of the underlying *manifold* over which the integrals are taken, and then utilize C^1-triangulations. (The best known counter-example, Schwarz' surface, what may go wrong with C^0 triangulations may be found in [8], vol. I, page 479).

References

[1] E. Acosts and C. Delgado, *Fréchet vs. Carathéodory, Amer. Math. Monthly, vol.* **101** *no.4, 1994, pp.332 - 338.*

[2] D. Hughes-Hallet, W. McCallum, et. al., *Multivariable Calculus*, Wiley, 1996.

[3] M. Kawski, **Limits and zooming: From continuity to Stokes' theorem**, (book under preparation).

[4] M. Kawski, MAPLE worksheets and related materials
URL: `http://math.la.asu.edu/indexpages/mat272/MAPLE.html`
and via anonymous ftp at `calculus.la.asu.edu`,
in the directories `/pub/kawski/sprg96/mat272/MAPLE`
and `/pub/kawski/sprg97/mat272/MAPLE`.

[5] P. Lax, Keynote address, 4^{th} int. Conf. Teaching of Mathematics, San Jose, CA, 1995.

[6] D. Lomen and D. Lovelock, The *ARIZONA SOFTWARE*, `http://math.arizona.edu/ software`.

[7] E. Purcell, *Electricity and Magnetism*, Berkeley Physics Course, McGraw-Hill, 1963.

[8] M. Spivak, *A comprehensive introduction to differential geometry*, Houston, 1970.

[9] K. Stroyan, *The Calculus: The Language of Change* Academic Press, 1997.

[10] David O'Tall, *Making research in mathematics education relevant to research mathematicians*, Joint Ann. Meetg. AMS and MAA, San Diego, 1997.