Connecting Dynamic Geometry and Analytic Geometry: A Shrinking Circle and A Shrinking Sphere

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Abstract

A limit problem was mentioned from James Stewart Calculus textbook (see [1]) and was extended by [2]. In this paper, we further extend the two dimensional results mentioned by [2] and we extend the results to three dimensions. It is indisputable that dynamic geometry software have contributed greatly when making conjectures on 2D and 3D Analytic Geometry.

1 Introduction

There is an interesting limit problem from the Calculus textbook by J. Stewart (see [1]). Consider the Figure 1 below which shows the circle C_1 of $(x-1)^2 + y^2 = 1$ and a shrinking circle C_2 of radius r, whose center is at the origin. Let P be the point $(0, r)$, and Q be the upper point of intersection between these two above. If we let R be the $x - intercept$ of the line PQ. It can be shown that R goes to $(4, 0)$ as C_2 shrinks to the origin, or as $r \to 0^+$.

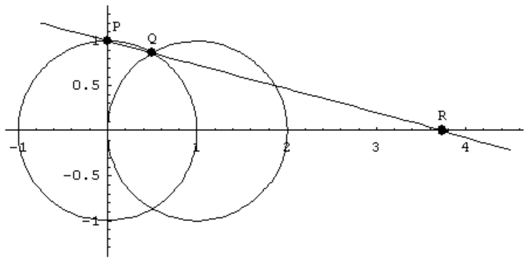


Figure 1: From Stewart Calculus Book

It was summarized by [2] that the limit R varies when the fixed curve C_1 is replaced by the following various curves (see Table 1).

Curve	Limit of R	Curvature at Origin
Line $y = ax$	0	0
Circle $(x - r)^2 + y^2 = r^2$	4r	$\frac{1}{r}$
Ellipse $\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{4b^2}{a}$	$\frac{a}{b^2}$
Hyperbola $\frac{(x+a)^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{4b^2}{a}$	$\frac{a}{b^2}$
Circle $(x - a)^2 + (y - b)^2 = a^2 + b^2$	0	$\frac{1}{\sqrt{a^2+b^2}}$

Table 1. A Summary from Robert Kreczner.

In this paper, we will extend the results mentioned in Table 1 to ones in three dimensions. But first we show the limit R goes to $(0, 0)$ (as $r \to 0^+$) when the figure C_1 is not symmetric to the $x - axis$.

Throughout this paper, we consider the intersection between the shrinking circle (or sphere) and the fixed curve (or surface) above $x - axis$ (or $xy - plane$).

2 When the center of the fixed circle is not on the $x-axis$

2.1 Exploration and Conjectures

We consider C_1 to be the circle $(x - c)^2 + (y - d)^2 = c^2 + d^2$ and C_2 to be the shrinking circle of $x^2 + y^2 = r^2$ as $r \to 0^+$. We use ClassPad Manager Version 3 (see [3]) to demonstrate how the animations and scatter plot lead us to conjecture that limit R goes to 0 as $r \to 0^+$.

The following sequence of plots (Figures 2 to 5 below) shows as C_2 shrinks to the origin when C_1 is the fixed circle $(x - 0.4)^2 + (y - 0.5)^2 = (0.4)^2 + (0.5)^2$, the point R (denoted by G below) goes to the right ($x > 0$) and later approaches to origin from the left ($x < 0$).

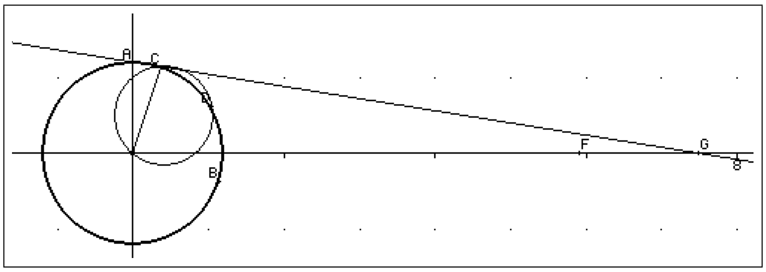


Figure 2: When C_2 is $x^2 + y^2 = 2$.

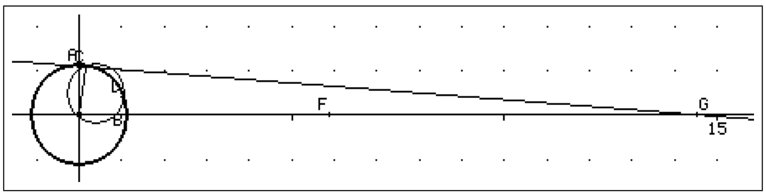


Figure 3: When C_2 is $x^2 + y^2 = 1.23$.

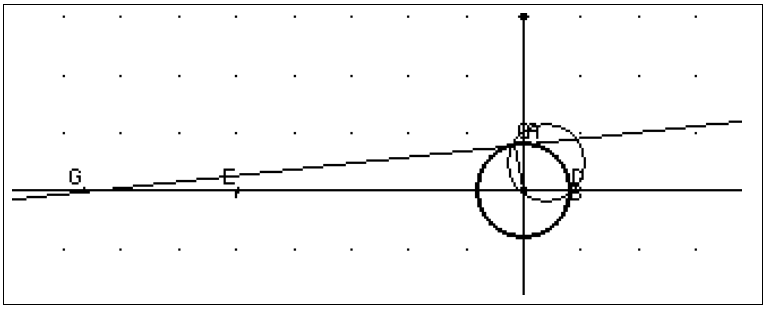


Figure 4: When C_2 is $x^2 + y^2 = 0.65$.

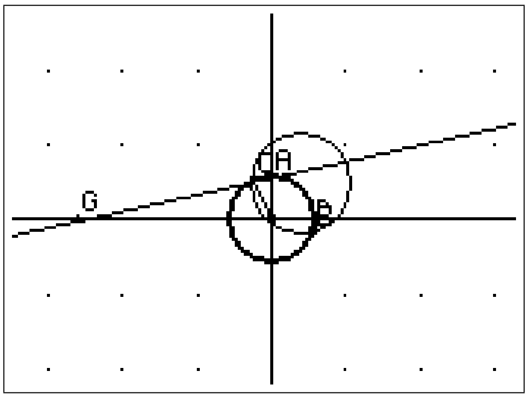


Figure 5: When C_2 is $x^2 + y^2 = 0.32$.

We collect the tabular data for R (or G as shown on Figures 2-5 above) as follow:

x	y
7.483282	0
9.089335	0
11.26514	0
14.51749	0
20.09051	0
32.19602	0
80.29126	0
-159.622	0
-39.5420	0
-22.3253	0
-15.3994	0
-11.6428	0
-9.27488	0
-7.63923	0
-6.43740	0
-5.51408	0
-4.78043	0
-4.18194	0
-3.68329	0
-2.89706	0
-2.58061	0
-2.30229	0
-2.05529	0
-1.83440	0
-1.63551	0
-1.45537	0
-1.29134	0
-1.14130	0
-1.00348	0
-0.87642	0
-0.75890	0
-0.64987	0
-0.54847	0
-0.45393	0
-0.36561	0
-0.28295	0
-0.20546	0
-0.13272	0
-0.06434	0

Table 2: Where does R go?

Notice from Table 2 that the $x - value$ of R goes to a positive large number and then to a negative large number and finally approaches to $(0, 0)$. We demonstrate this obervation again by using the following scatter plot (see Figure 6 below):

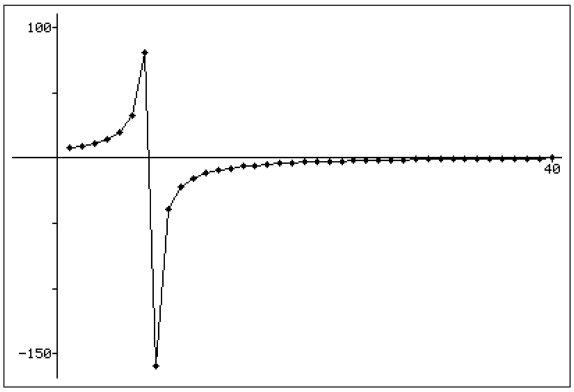


Figure 6: Scatter plot of the limit R.

Therefore, our conjecture is either R goes to positive infinity, negative infinity or $(0,0)$. Let's further analyze this problem below.

2.2 Further Analysis

2.2.1 Using Rectangular Coordinate

Assume C_1 and C_2 are represented by

$$\begin{aligned} x^2+y^2 &= r^2 \text{ and} && (1)\\ (x-c)^2+(y-d)^2 &= c^2+d^2 && (2) \end{aligned}$$

respectively. Let $P=(0,r)$ be on C_1, where $r>0$ and $Q=(x_2,y_2)$ be the intersection between C_1 and C_2 in the first or second quadrant. We note the $x-intercept$ can be written as

$$x=-\frac{r}{m}=-\frac{rx_2}{y_2-r}, \tag{3}$$

where $m=\dfrac{y_2-r}{x_2}$.

To find Q, we solve equations (1) and (2) simultaneously, and take only the intersection above the $x-axis$, we get the following x_2 and y_2, which both Maple 10 (see [6]) and Scientific WorkPlace 5.5 (SWP 5.5) (see [5]), which uses MuPAD (see [8]) as its computation engine agree.

$$\begin{aligned} x_2 &= \frac{1}{2c}\left(r^2-d\frac{dr^2+cr\sqrt{4c^2+4d^2-r^2}}{c^2+d^2}\right), \\ y_2 &= \frac{1}{2}\frac{dr^2+cr\sqrt{4c^2+4d^2-r^2}}{c^2+d^2} \end{aligned} \tag{4}$$

By substituting x_2 and y_2 into equation (3), we find the $x-intercept$ of PQ to be

$$\frac{cr^2-dr\sqrt{4c^2+4d^2-r^2}}{2c^2-dr-c\sqrt{4c^2+4d^2-r^2}+2d^2} \tag{5}$$

Finally, we let $r\to 0^+$ and compute symbolically to get

$$\lim_{r\to 0}\left(\frac{cr^2-dr\sqrt{4c^2+4d^2-r^2}}{2c^2-dr-c\sqrt{4c^2+4d^2-r^2}+2d^2}\right)=0 \tag{6}$$

2.2.2 Using Polar Coordinate

Finding the intersection(s) for two curves may not be trivial, in addition to the rectangular coordinate method we described above, we use polar coordinate to find the intersection(s) between C_1 and C_2 and later show that the limit goes to 0 as well. We substitute $x = r\cos\theta$ and $y = r\sin\theta$ into $(x-c)^2+(y-d)^2 = c^2+d^2$ for the curve C_2 and solve the following equation for θ,

$$(r\cos\theta - c)^2 + (r\sin\theta - d)^2 = c^2 + d^2. \tag{7}$$

Surprisingly, we get the following three answers when different computation engines are used.

1. When Maple 10 is used, we get

$$\left[\arctan\left(\frac{r - \frac{\left(r\,c + \sqrt{4\,d^4 - d^2\,r^2 + 4\,c^2\,d^2}\,\right)c}{c^2+d^2}}{d}, \frac{r\,c + \sqrt{4\,d^4 - d^2\,r^2 + 4\,c^2\,d^2}}{c^2+d^2}\right), \arctan\left(\frac{r - \frac{\left(r\,c - \sqrt{4\,d^4 - d^2\,r^2 + 4\,c^2\,d^2}\,\right)c}{c^2+d^2}}{d}, \frac{r\,c - \sqrt{4\,d^4 - d^2\,r^2 + 4\,c^2\,d^2}}{c^2+d^2}\right)\right]$$

2. When ClassPad Manager Version 3 is used, we get

$$\mathrm{solve}(r^2-2\cdot c\cdot r\cdot \cos(t)-2\cdot d\cdot r\cdot \sin(t)=0,t)$$

$$\left\{t=2\cdot \tan^{-1}\left(\frac{2\cdot d}{2\cdot c+r}-\frac{\sqrt{4\cdot c^2+4\cdot d^2-r^2}}{2\cdot c+r}\right)+2\cdot\pi\cdot \mathrm{constn}(1),\right.$$

$$\left. t=2\cdot \tan^{-1}\left(\frac{2\cdot d}{2\cdot c+r}+\frac{\sqrt{4\cdot c^2+4\cdot d^2-r^2}}{2\cdot c+r}\right)+2\cdot\pi\cdot \mathrm{constn}(2)\right\}$$

3. When SWP 5.5 is used, the answer is too long to be listed here.

We use the angle t obtained from CP for demonstration here. We recall that the $x-intercept$ of PQ is

$$x = -\frac{r}{m} = -\frac{rx_2}{y_2 - r} = \frac{r^2(\cos\theta)}{r\sin\theta - r}, \tag{8}$$

we notice that we have two choices of angles here and we leave it to the reader to verify that we will take only the following angle into consideration.

We define the angle θ to be the function

$$f(c,d,r) = 2\arctan\left(\frac{2d}{2c+r} + \frac{\sqrt{4c^2+4d^2-r^2}}{2c+r}\right). \tag{9}$$

We find some interesting observations below:

Remark 1 *If we take $c = d$, and both c and d are non-zero, consider the angle HOC or complement of the angle HOC below (where O is the origin), which is our $f(c,c,r)$ or $f(d,d,r)$. What is the limit of the angle HOC or the complement of the angle HOC when $r \to 0^+$?*

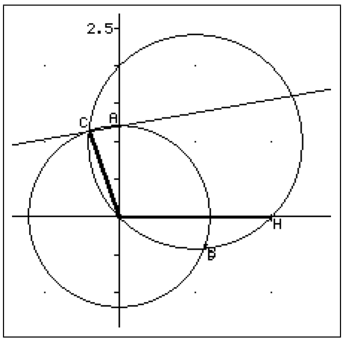


Figure 7: When $c = d$.

It is interesting to see that ClassPad Manager provides us the following conjecture:

53.77600067
53.31903438
52.86260041
52.40666803
51.95120681
51.4961866
51.0415775
50.58734986
50.13347423
49.67992138
49.22666226
48.77366798
48.32090982
47.86835917
47.41598755
46.96376658
46.51166795
46.05966346
45.60772491

Table 3 : Partial list of the complement of the angle HOC as $r \to 0^+$.

This can be verified by observing

$$\lim_{r \to 0^+} f(c,c,r) = 2\arctan(1+\sqrt{2}) = 2.356\,2 \text{ radian} = 135 \text{ degree.} \tag{10}$$

Remark 2 *When $c \neq d$, then*

$$\lim_{r \to 0^+} f(c,d,r) = 2\arctan\left(\frac{d+\sqrt{c^2+d^2}}{c}\right), \tag{11}$$

which we obtain from Maple 10.

Finally, by substituting (9) into (8), it is not hard to see the following limit to be 0 as expected.

$$\lim_{r \to 0}\left(\frac{r\left[\cos\left(2\arctan\left(\frac{2d}{2c+r}-\frac{\sqrt{4c^2+4d^2-r^2}}{2c+d}\right)\right)\right]}{\left[\sin\left(2\arctan\left(\frac{2d}{2c+r}-\frac{\sqrt{4c^2+4d^2-r^2}}{2c+d}\right)\right)-1\right]}\right) \tag{12}$$

The following example mentioned in [2] illustrates the fact that finding the limit for $\lim_{r\to 0^+}\left(-\dfrac{rx_2}{y_2-r}\right)$ causes some problems for the Mathematica (see [7]). We show here that either rectangular or polar coordinate method is used to find such limit, same problem persists for SWP 5.5 and Maple 10.

Example 3 *Let C_1 be $y^2 = 2ax$ and C_2 be the shrinking circle $x^2 + y^2 = r^2$ as $r \to 0^+$. It can be shown that the curvature for C_1 at the origin is $\frac{1}{a}$. Therefore, according to the Table 1, the limit R will be $(4a, 0)$. However, we note the followings:*

1. *When rectangular coordinate method is used in finding $\lim_{r\to 0^+}\left(-\frac{rx_2}{y_2-r}\right)$, we are calculating*
$$\lim_{r\to 0^+}\left(\frac{r(a-\sqrt{a^2+r^2}}{\sqrt{-2a^2+2a\sqrt{a^2+r^2}}-r}\right), \tag{13}$$
we need to set a to be a constant number before we compute the limit; otherwise, we achieve the wrong answer 0 from SWP 5.5, Maple 10 and Mathematica as mentioned in [2]. For example, if $a = \pi$, then the limit will be $(4\pi, 0)$.

2. *When polar coordinate method is used in finding $\lim_{r\to 0^+}\left(-\frac{rx_2}{y_2-r}\right)$, we are calculating*
$$\lim_{r\to 0^+}\left(\frac{r^2\cos\left(\pi-\arccos\left(\frac{a-\sqrt{a^2+r^2}}{r}\right)\right)}{r\sin\left(\pi-\arccos\left(\frac{a-\sqrt{a^2+r^2}}{r}\right)\right)-r}\right), \tag{14}$$
we also need to set a to be a constant number before we compute the limit; otherwise, we achieve the wrong answer 0 from SWP 5.5 and Maple 10. For example, if $a = \pi$, then the limit will be $(4\pi, 0)$.

Theorem 4 *If the curvature circle C_3 of a curve C_1 at origin is described by the equation $(x-c)^2 + (y-d)^2 = c^2 + d^2$, then as the circle $x^2 + y^2 = r^2$ shrinks to 0 as $r \to 0^+$, the point R approaches origin.*

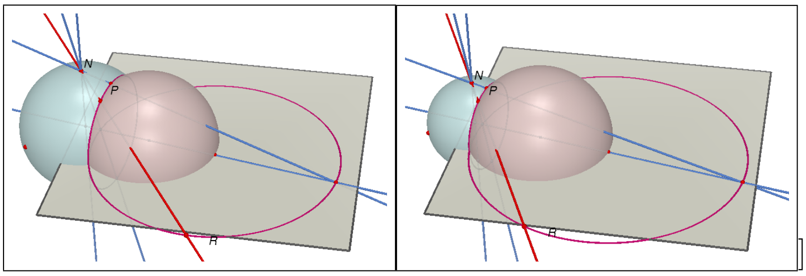
Proof. Since the limit (6) or (12) does not depend on where the center or radius is, the result follows directly by applying either Rectangular or Polar Coordinate method above. ■

3 Three Dimensional Cases

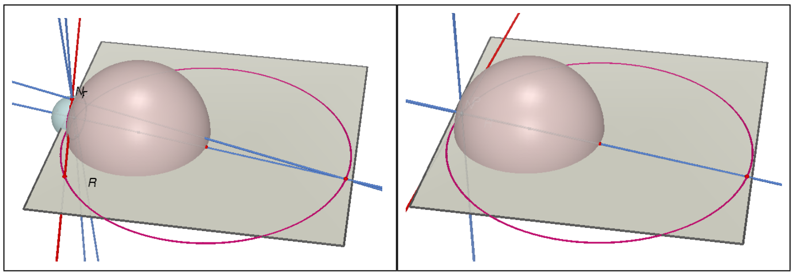
We consider the sphere S_1 of $(x-1)^2 + y^2 + z^2 = 1$ and the sphere S_2 of $x^2 + y^2 + z^2 = r^2$. Let P be the point $(0, 0, r)$, and Q be any point of intersection between these two spheres above the xy plane. Let R be the intersection between the line PQ and the xy plane. What is the locus of R as the sphere S_2 shrinks to the origin or as $r \to 0^+$?

3.1 Exploration

We conjecture from the following sequence of animations (done by Cabri 3D, see [4]) that the for each fixed r, the point R will satisfy an elliptic curve. We also conjecture that locus of R will be also an elliptic curve as $r \to 0^+$.



Figures 8 and 9: When the sphere is shrinking



Figures 10 and 11: When the sphere is shrinking even further

3.2 Symbolic Analysis

We consider $S_1 : (x - A)^2 + y^2 + z^2 = A^2$ and $S_2 : x^2 + y^2 + z^2 = r^2$, we find the intersection to be

$$\left[x = \frac{1}{2A} r^2, z = \frac{1}{2A}\sqrt{4A^2r^2 - 4A^2y^2 - r^4}\right] \tag{15}$$

Let $P = (0, 0, r)$ and $Q = (\frac{1}{2A}r^2, t, \frac{1}{2A}\sqrt{4A^2r^2 - 4A^2y^2 - r^4})$, which is the intersection between these two spheres above xy plane. Then we parametrize the line equation PQ as follows

$$\begin{aligned} x &= a \cdot s & (16)\\ y &= b \cdot s & (17)\\ z &= r + c \cdot s, & (18) \end{aligned}$$

where $a = \frac{1}{2A}r^2, b = t$, and $c = \frac{1}{2A}\sqrt{4A^2r^2 - 4A^2y^2 - r^4} - r$. We set $z = 0$ to get

$$s = \frac{2rA}{-\sqrt{4r^2A^2 - 4t^2A^2 - r^4} + 2rA}. \tag{19}$$

This gives the project of PQ onto the xy plane to be as follows:

$$\begin{aligned} x &= \frac{r^3}{-\sqrt{4r^2A^2 - 4t^2A^2 - r^4} + 2rA} & (20)\\ y &= \frac{2t \cdot r \cdot A}{-\sqrt{4r^2A^2 - 4t^2A^2 - r^4} + 2rA}. & (21) \end{aligned}$$

We set

$$L = (x - 2A)^2 + y^2 \tag{22}$$

We substitute above x and y into L and compute symbolically with Maple 10. We leave it to the reader to verify that L satisfies an equation of a circle. In addition, we get the following limit by using Maple 10:

$$\lim_{r \to 0} \left((x - 2A)^2 + y^2 \right) = (2A)^2 , \tag{23}$$

which means the point $(4A, 0)$ is on the locus $(x - 2A)^2 + y^2 = (2A)^2$. We note that this is consistent with two dimensional case. We further make the following observations fpr

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2 \text{ and} && (24)\\ (x-2)^2 + y^2 + z^2 &= 4. && (25) \end{aligned}$$

Remark 5 *By looking at the intersection Q above, we notice that we touch the top of the intersection (largest z.value) for Q if $y = 0$.*

Remark 6 *For a fixed r, if we vary y, the projection of PQ onto the xy plane satisfies the equation $L = (x - 2A)^2 + y^2$.*

Remark 7 *When r gets closer to $0^+, L$ is getting closer to* 16.

We summarize the above observations in the following Tables 4-6.

r	A	t	L
1	2	0	15.00000274
1	2	0.1	15.00000105
1	2	0.2	15.00000032
1	2	0.3	14.99999950

Table 4: When $r = 1$

r	A	t	L
0.5	2	0	15.75019366
0.5	2	0.1	15.74999118
0.5	2	0.2	15.75000073
0.5	2	0.3	15.74999998

Table 5: When $r = 0.5$

r	A	t	L
0.01	2	0	0
0.01	2	0.1	$15.99990000 + 0.1206376334e - 9 * I$
0.01	2	0.2	$15.99990000 - 0.7004858473e - 10 * I$
0.01	2	0.3	$15.99990001 - 0.6674664446e - 9 * I$

Table 6: When $r = 0.01$

We notice from Table 6 that it is incorrect (from Maple 10) to have $L = 0$ when $r = 0.01$, $R = 2$ and $t = 0$, it is supposed to be a number close to 16. Also the complex number I appears in the above table which might have been due to numerical limitations from Maple 10. We sketch $x^2 + y^2 + z^2 = 1$, $(x-2)^2 + y^2 + z^2 = 4$, the vector PQ where $P = (0, 0, 1)$ and Q is the top intersection $(0.25, 0, 0.9682458365)$, and the projected circle $(x-4)^2 + y^2 = 15$ below

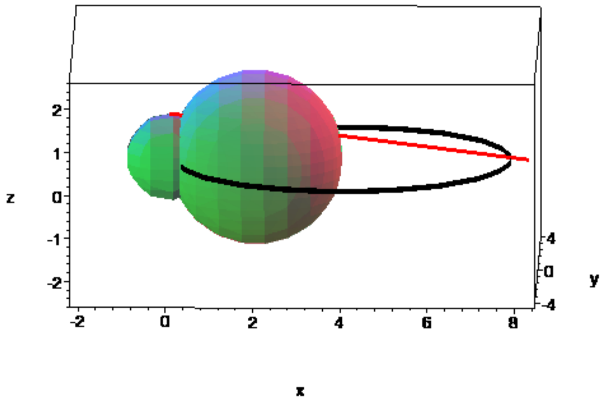


Figure 12: A projection done with Maple 10

Theorem 8 *Let S_1 be of the surface whose Gaussian curvature is same as $(x-A)^2+y^2+z^2 = A^2$ near origin and S_2 be the shrinking sphere $x^2 + y^2 + z^2 = r^2$ as $r \to 0^+$. Let P be the point $(0, 0, r)$, and Q be any point of intersection between these two spheres above the xy plane. Let R be the intersection between the line PQ and the xy plane. Then the locus of R, when sphere S_2 shrinks to the origin, is*

$$(x - 2A)^2 + y^2 = (2A)^2 \,. \tag{26}$$

Similarly, if S_1 is of the surface whose Gaussian curvature is same as $\dfrac{(x-a)^2}{a^2} + \dfrac{y^2}{b^2} + z^2 = 1$ at the origin, then the locus of R will be

$$\frac{(x-2a)^2}{(2a)^2} + \frac{y^2}{b^2} = 1, \tag{27}$$

as $r \to 0^+$. If S_1 is of the surface whose Gaussian curvature is same as $\dfrac{(x+a)^2}{a^2} - \dfrac{y^2}{b^2} - z^2 = 1$ at the origin, then the locus of R will be

$$\frac{(x+2a)^2}{(2a)^2} - \frac{y^2}{b^2} = 1, \tag{28}$$

$r \to 0^+$.

Proof. We prove only for the case of $(x-A)^2 + y^2 + z^2 = A^2$, the other two cases follow similarly.

We follow the idea that is mentioned in [2] when proving two dimensional case. Let $\epsilon > 0$ be any real number, and let S_3 be of the sphere of $(x-A)^2 + y^2 + z^2 = A^2$and S_4 be sphere of $(x-(A+\epsilon))^2 + y^2 + z^2 = (A+\epsilon)^2$. Let U and V be the intersection between the line PQ and the xy plane with the surfaces S_3 and S_4 respectively. Then since we assume S_1 has the same

Gaussian curvature near the origin as S_3 , we observe R lies between two circles where U and V belong to. Since the loci of U and V are

$$(x-2A)^2+y^2 = (2A)^2 \text{ and} \quad (29)$$
$$(x-(2A+\epsilon))^2+y^2 = (2A+\epsilon)^2 \quad (30)$$

respectively as $r \to 0^+$. We conclude that the locus of R has to be between the above two equations, and since $\epsilon > 0$, the limit of R is $(x-2A)^2+y^2=(2A)^2$ as $r \to 0^+$. ■

Theorem 9 *Let S_1 be of the surface whose Gaussian curvature is same as $(x-A)^2+(y-b)^2+z^2=A^2+b^2$ near origin, and S_2 be the shrinking sphere $x^2+y^2+z^2=r^2$ as $r \to 0^+$.Let P be the point $(0,0,r)$, and Q be a point of intersection between these two spheres above the xy plane. Let R be the intersection of the line PQ and the xy plane. Then the locus of R is $(0,0,0)$ when sphere S_2 shrinks to the origin.*

Proof. We project both $(x-A)^2+(y-b)^2+z^2=A^2+b^2$ and $x^2+y^2+z^2=r^2$ onto the xz plane, the result follows directly from the two dimensional limits (6) and (12). ■

4 Conclusion

The orginal problem from [1] is interesting and could have been solved by finding the limit algebraically. However, the dynamic geometry software such as ClassPad V3 and Cabri 3D have made the animations in 2D and 3D respectively much more interesting and provide the basis for analytical proofs. The paper also shows that the algebraic and numerical computations would have not been possible if we don't have a computer algebra system to find solutions symboically, numerically and graphically; although we note that it is important to use different Computer Algebra Systems to check if numeric or symbolic answers are meaningful and usable. Author predicts that the future integration between a dynamic geometry software and a computer algebra system will further expand our ability to tackle many more interesting, realistic and challenging problems.

4.1 Acknowledgment

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References

[1] James Stewart, *Calculus*; Concepts and Contexts, 2nd edition, Brooks/Cole, 2001.

[2] Kreczner, Robert, *AN INTERESTING PROBLEM FROM CALCULUS AND MUCH MORE*, ICTCM 8, Electronic Proceeding (http://archives.math.utk.edu/ICTCM/EP-8/C11/pdf/paper.pdf).
Quotated URLs:

[3] ClassPad Manager, a product of CASIO Computer Ltd. http://classpad.net or http://classpad.org.

[4] Cabri 3D, a product of CABRILOG, http://www.cabri.com/v2/pages/en/index.php.

[5] Scientific WorkPlace, a product of MacKichan Software Inc. http://www.mackichan.com/.

[6] Maple, a product of Maplesoft, http://maplesoft.com.

[7] Mathematica, a product of Wolfram Research, Inc., http://wolfram.com

[8] MuPad, a product of SciFace Software GmbH & Co. KG, http://www.sciface.com/.