Inspiring Students to Study and Learn Mathematics Using Technology

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Abstract

In this paper, we focus on the advantages of studying mathematics from analytical, graphical and geometric perspectives. Using Casio's ClassPad 300 ([1]), we present activities on problems involving abstract concepts and real life applications. These activities viewed from the analytical, graphical, tabular and geometric points of view, promote creative ways to facilitate the learning of mathematics that inspire students with varying levels of ability. Consequently, students develop a profound appreciation and a deeper understanding of mathematics.

Introduction

It is a fact that more and more students are showing less interest in studying mathematics.
For many students, learning mathematics is often the memorization of mechanical procedures. This contributes directly to the development of poor problem solving and analytical skills when students enter college or university. Consequently, many mathematics topics are taught in high school and university repeatedly.

How can teachers motivate students to study mathematics? How can teachers make mathematics more accessible to the weaker students and more challenging to the good students? (See [2].)

Technology can provide mechanisms to sustain assistance to mathematics teachers in their use of technology to implement mathematics education reforms in their classes. Technology enables mathematics education reform, but it is not reform per se (see [3]). We must provide mathematics teachers extended opportunities to experience and do mathematics in an environment supported by diverse technologies (see [4]). The heart of our approach is the development of mathematical power-- understanding, using, and appreciating mathematics. Our interest is in empowering teachers through the use of technology in mathematics exploration, open-ended problem solving, interpreting mathematics, developing understanding, and communicating about mathematics (see [5]).

Most recently, the emergence of a powerful technological tool which incorporates the power of a graphics calculator with a computer algebra system (CAS) and a geometry package farther pronounces the role of technology in making mathematical concepts more concrete and meaningful to the learner.

The advent of Casio's Class Pad 300, which incorporates a CAS and geometry capability, allows users to explore many mathematical concepts simultaneously- graphically, numerically, symbolically and geometrically. It comes at an appropriate time when mathematics educators are in search of more innovative ways to inspire our students to love and appreciate mathematics.

Teaching mathematics from an analytical, graphical and geometric perspective

In this part of the paper, we present actual activities which emphasize studying a problem from various angles. (We illustrate the activities using *Casio's Classpad 300*, which has a unique feature, called an e-activity menu, which allows one to functionally link its different applications with one another- numerical and algebraic calculations, graphical as well as tabular presentations and dynamic geometry.)

1. A geometric perspective together with a graphical approach facilitates a clearer understanding of analytical concepts. Theorems, which are oftentimes considered as abstract notions by students become more accessible. The implications of the theorems can be visualized and easily understood.

Take for instance the following application of the Mean Value Theorem.

Example. Show that the Mean Value Theorem holds for $f(x) = x^3 + x^2 - x$ on the interval [-2,1]

Solution:
We first construct the secant line passing through the points $(-2, f(-2))$ and $(1, f(1))$ (see Figure 1). The equation of the secant line is $y = x$.
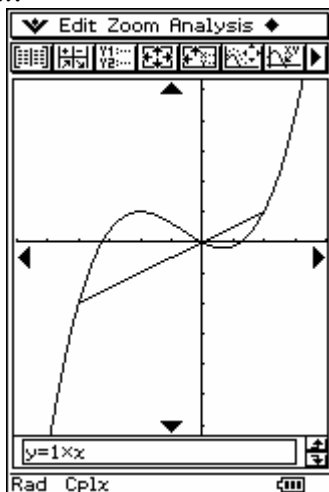

Figure 1

Analytically, we can calculate the existence of the numbers in the interval (-2,1) satisfying the mean value theorem by solving for x in $f'(x) = 1$. We will obtain the approximations $x = -1.2$ and $x = 0.54$ (see Figure 2).
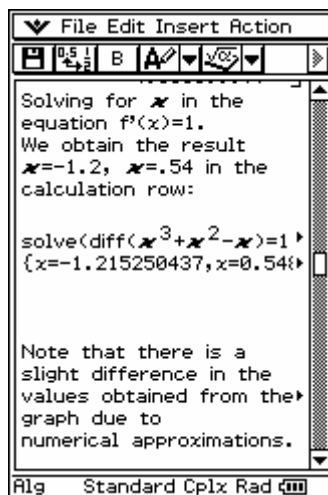
Figure 2

Using a geometric approach, the existence of these numbers in the interval (-2,1) can be visualized. We draw $f(x)$ in the geometry window and construct the tangent line to the curve.

An animation routine in the geometry window allows the student to conjecture that the tangent line assumes a slope of approximately 1 twice in the whole cycle (see Figures 3, 4 and 5 below).
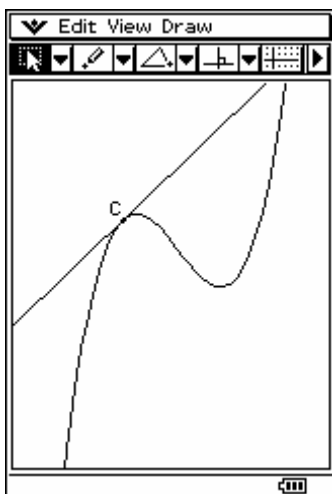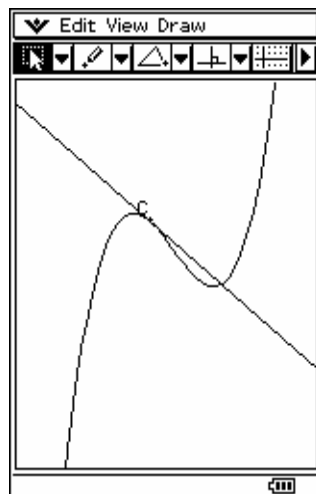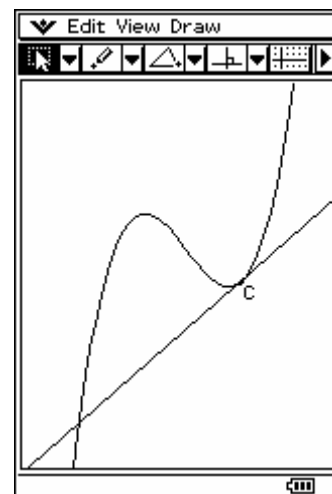

Figure 3   **Figure** 4   **Figure** 5

A tables of slopes of the tangent line as the line is animated at various points on the curve may be generated which will strengthen the observation that the slope is approximately 1 at $x = -1.2$ and $x = 0.5$ (see Figures 6 and 7).
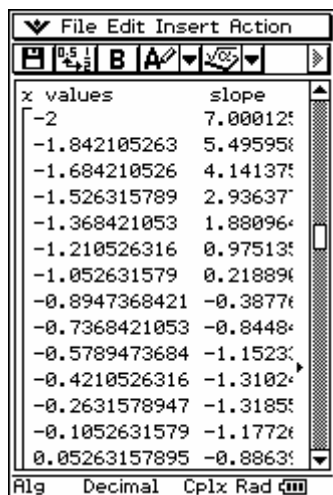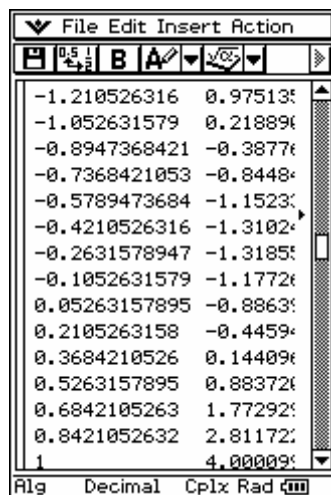
x values	slope
-2	7.000125
-1.842105263	5.495958
-1.684210526	4.141375
-1.526315789	2.936371
-1.368421053	1.880964
-1.210526316	0.975135
-1.052631579	0.218896
-0.8947368421	-0.38776
-0.7368421053	-0.84484
-0.5789473684	-1.15233
-0.4210526316	-1.31024
-0.2631578947	-1.31855
-0.1052631579	-1.17726
0.05263157895	-0.88639

Figure 6

-1.210526316	0.975135
-1.052631579	0.218896
-0.8947368421	-0.38776
-0.7368421053	-0.84484
-0.5789473684	-1.15233
-0.4210526316	-1.31024
-0.2631578947	-1.31855
-0.1052631579	-1.17726
0.05263157895	-0.88639
0.2105263158	-0.44594
0.3684210526	0.144096
0.5263157895	0.883726
0.6842105263	1.772929
0.8421052632	2.811722
1	4.000099

Figure 7

In the Graph window, we can visualize that the slope of the tangent lines at $x = -1.2$ and $x = 0.54$ is approximately 1 (see Figures 8 and 9).

dy/dx=0.9949
xc=-1.214285 ▼ yc=0.8983236
y=0.9948800006×x+2.106
Rad Cplx

Figure 8

dy/dx=0.9493
xc=0.538961 ▼ yc=-0.091925
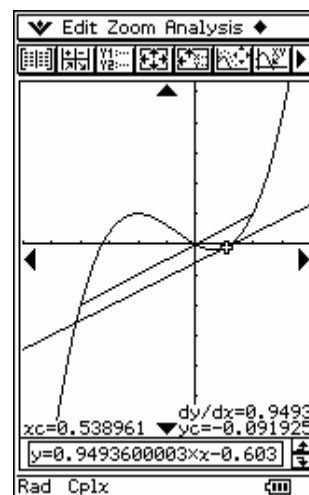y=0.9493600003×x-0.603
Rad Cplx
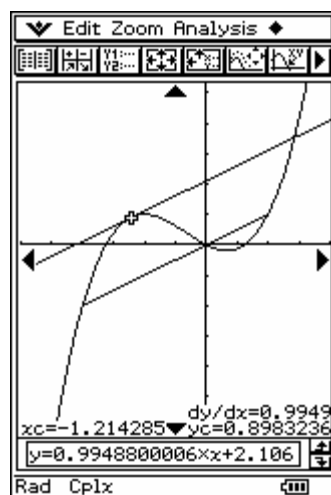
Figure 9

2. Mathematical formulas become more significant to students when their derivation is obtained using a combination of graphical, tabular and geometric approach.

Example: Show $\dfrac{d}{dx}(\sin x) = \cos x$.

Solution:
Our approach is to graph the sine function in the Cartesian plane. Next, we select a point on the curve and construct a tangent line as follows (see Figure 10):
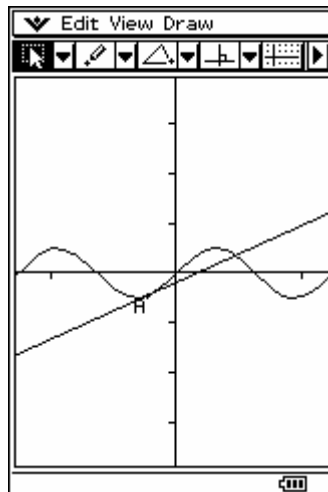
Figure 10

Using animation, we can visualize the line as it becomes tangent to points throughout the sine curve. A table can be generated (see Figures 11 and 12 below) which reflects the slopes of the tangent line as the tangent line moves throughout the animation, and also the coordinates of the points of tangency.

Figure 11

Figure 12

If we drag both the columns of *x*- values and the slopes into the graph screen of the calculator, the graph of $y = \cos x$ is generated. On the other hand, if we repeat the process for both the columns of *y*-values and the slopes, the unit circle will be generated (see Figure 13).
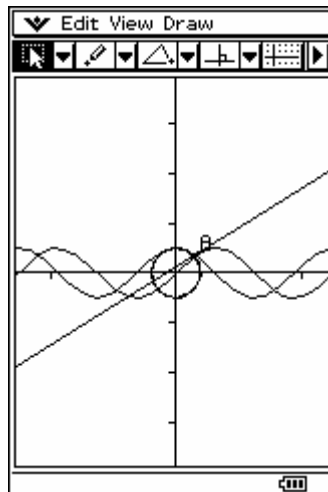
Figure 13

Through the exercise, the student can visualize the connection between the slopes of the tangent line at any point on the curve, with the cosine function. This provides a clearer understanding of the notion of the derivative and the formula $\frac{d}{dx}(\sin x) = \cos x$. Moreover, the idea that the values of $x = \sin t$ and $y = \cos t$ are obtained from the unit circle is also given emphasis.

3. One way to encourage and inspire students to study mathematics is to provide them with opportunities to interpret the mathematics that appears every day in the world around them. Mathematical modeling of real-life applications can be given emphasis through a graphical and geometric approach.

Consider the following problem which pertains to rates of change of economic quantities such as cost, revenue and profit.

Example: A manufacturing company estimates that the cost in dollars of producing x items is $C(x) = \frac{1}{8} x^2 + 3x + 98$ where all units will be sold if price is given by $p(x) = \frac{1}{3} (75\text{-}x)$ dollars per unit.

a) Determine the production level that will maximize profit
b) Verify that at the point where profit is maximum, marginal revenue = marginal cost
c) How many units must the manufacturer sell to break-even?

Solution:
a) Profit is defined to be Revenue – Cost. Thus, the profit function, $P(x)$ is given to be $P(x) = R(x) - C(x)$ where $R(x) = x\, p(x)$. For this problem, we have
$$P(x) = x\, \frac{1}{3}(75\text{-}x) - (\frac{1}{8}x^2 + 3x + 98) = -\frac{11}{24}x^2 + 22x - 98.$$
In the graph window, the calculator returns the maximum point from the graph of the profit function $P(x) = -\frac{11}{24}x^2 + 22x - 98$ (see Figure 14). The maximum profit is attained when 24 items is sold, at the point where $P'(x) = 0$ (see Figure 15).
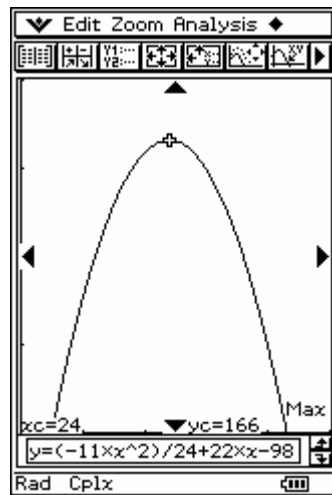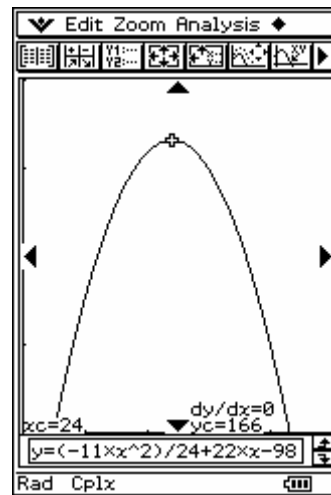
Figure 14   **Figure** 15

b) To explore the validity of one of the laws of economics, which states that at the point where profit is maximum, marginal revenue = marginal cost, we construct tangent lines to the revenue and cost functions when $x = 24$ (see Figure 16). The curves have parallel tangents at the production level $x = 24$ since marginal revenue equals marginal cost. In the graph windows (Figure 17) below, the slopes of the tangent lines, given by the derivatives are both approximately equal to 9.
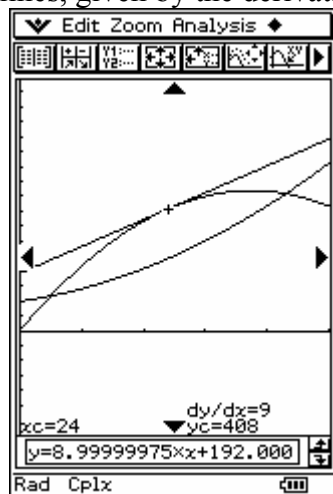
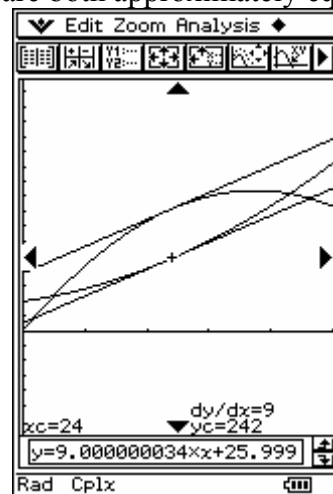.   **Figure** 16   **Figure** 17

c) The break even points are the points where revenue = cost or profit is zero. It can be explored in the graph window that these are the points of intersection of the revenue and cost functions, and at the same time the roots of the profit function. The break even points for the given problem occur at approximately $x = 5$ and $x = 43$ (see Figures 18 and 19).
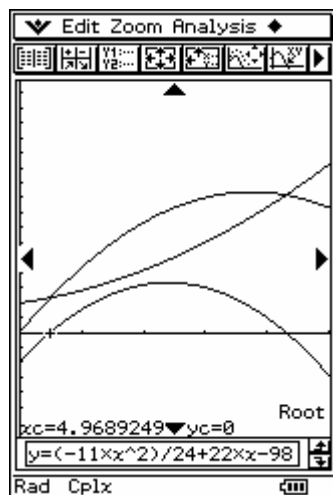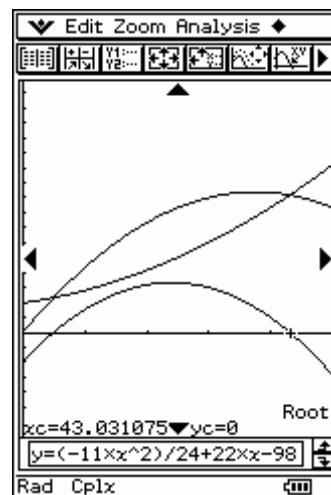
Figure 18   **Figure** 19

4. An exploratory approach to word problems encourages students to be more involved in problem solving. Moreover, when students are exposed to a variety of approaches to finding a solution, their skill in solving mathematical problems grows.

Example: Explore the area of an isosceles triangle with inscribed unit circle. Find the minimum and maximum area.

Solution:
By animating an isosceles triangle with inscribed unit circle in the geometry window (see Figures 20, 21 and 22), the students can explore and conjecture when the area will be minimum or maximum.
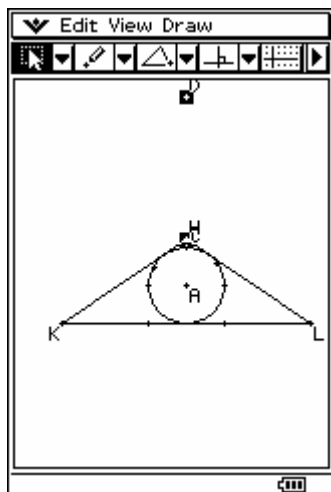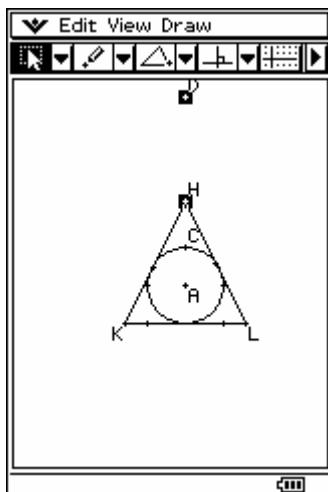

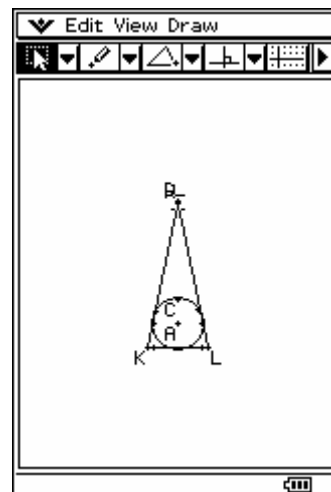**Figure** 20   **Figure** 21   **Figure** 22

As the triangle is animated, a table showing the areas of the triangle can be generated as follows (see Figure 23):

Figure 23

It can be verified from the table when the triangle attains a minimum area. For example, when the angle is approximately between 59 and 62 degrees, the triangle attains a minimum area.

Another approach to solve the problem would be to use the area of the triangle. It can be shown that the area of the triangle is given by the function

$$A(x) = \frac{1}{2} \frac{[1 + \dfrac{1}{\sin(\dfrac{x}{2})}]^2 \sin x}{(\cos(\dfrac{x}{2}))^2},$$

where x is the angle KHL. From the graph of the area function (see Figure 24), it can be predicted that the maximum area will occur at the vertical asymptotes of the function, $x = 0^0$ and $x = 180^0$ and the minimum area will occur at the lowest point on the graph, $x = 60^0$. The graph of the derivative (darker curve; see Figure 25) shows that at these points either $A'(x)$ does not exist or $A'(x) = 0$.
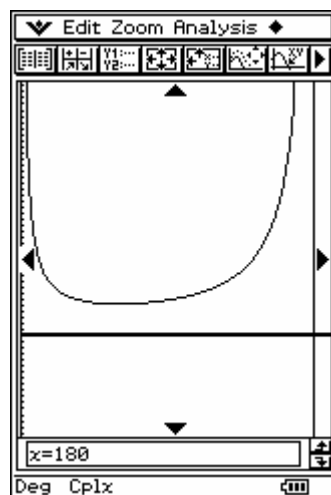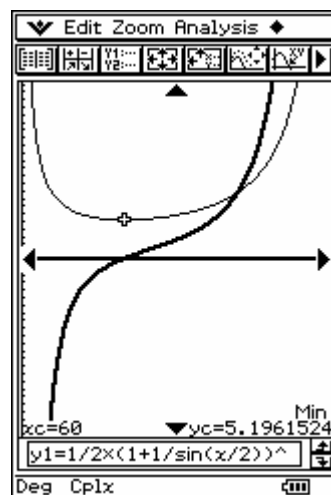

Figure 24

Figure 25

Conclusion

In this paper, we used four examples to demonstrate how technology can facilitate studying mathematical concepts from various perspectives. Abstract mathematical notions can be visualized by students by linking graphs, tables and geometric figures. Theorems and mathematical formulas can be made more interesting by using animations.

Students can appreciate the applications of mathematics by exploring real life problems and conjecture their mathematical implications.

The emergence of a technological tool like the Class Pad 300, with its computer algebra system (CAS) capability and a geometric functionality, provides teachers and students with new and innovative ways on how to study mathematics.

References

[1] ClassPad, a product by CASIO Computer Co.,Ltd. (http://www.casio.co.jp/edu_e/)

[2] Yang, W-C. *Making Mathematics More Accessible and Challenging with Technology.* To appear at Proceedings of the 18th Reunion Latinoamericana de Matematica Educativa (RELME 18, July 19-23, 2004, Chiapas, Mexico).

[3] Kaput, J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

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